National Technical University of Athens

Modified Gravity and Cosmology

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CANTATA

Cosmology and Astrophysics Network for Theoretical Advances and Training Actions

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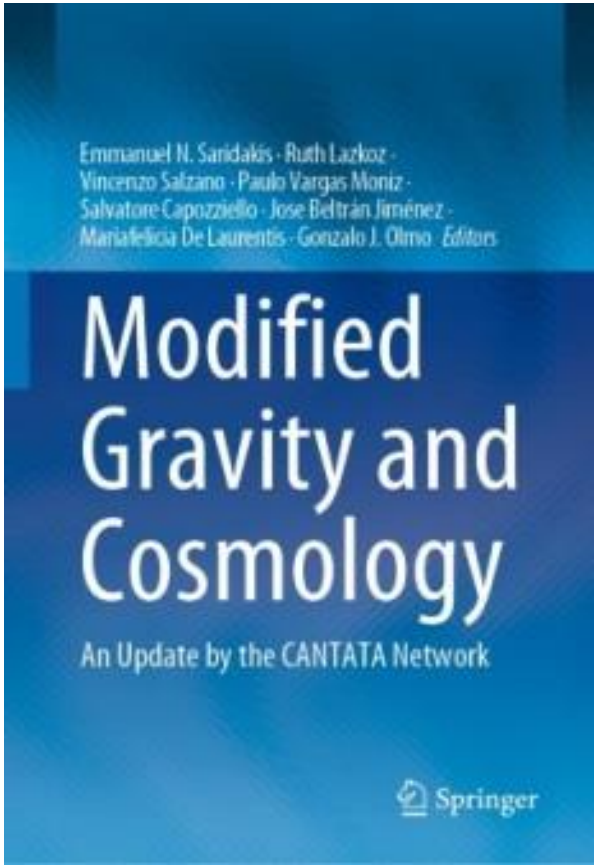
cost

EUROPEAN COOPERATION IN SCIENCE & TECHNOLOGY

Bibliography



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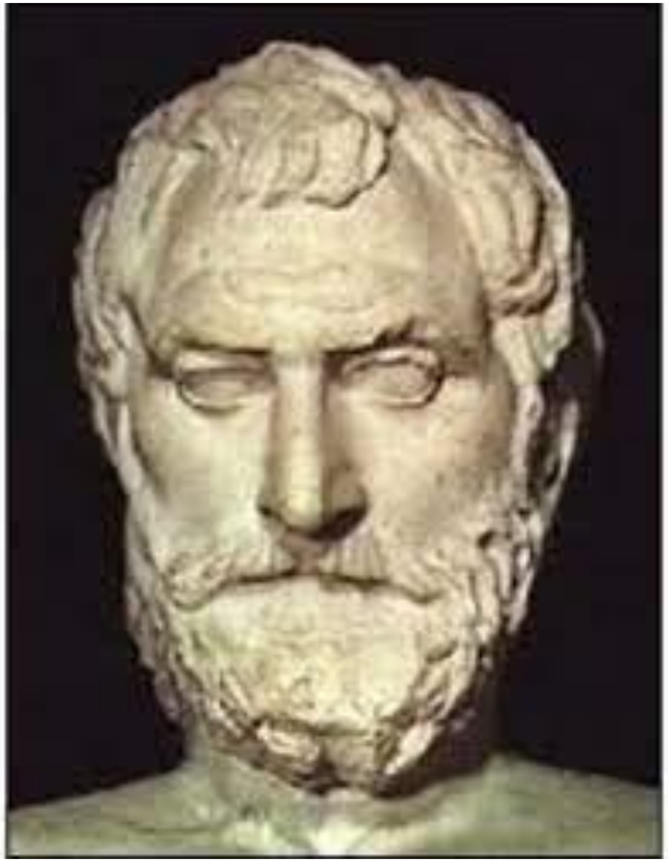
- Gravity is the most interesting interaction in Nature, and the one we know less about

- Gravity determines the Universe evolution: Cosmology

- Cosmology has become a precision science with an incredibly huge amount of data

Thales of Miletus 624 BC- 548 BC



Desrcibe Natute (Physis) without superstition, myth or religion, but with observation, experimentation, superposition and deduction. The term Physics appears.

Pythagoras of Samos 570 BC- 495 BC

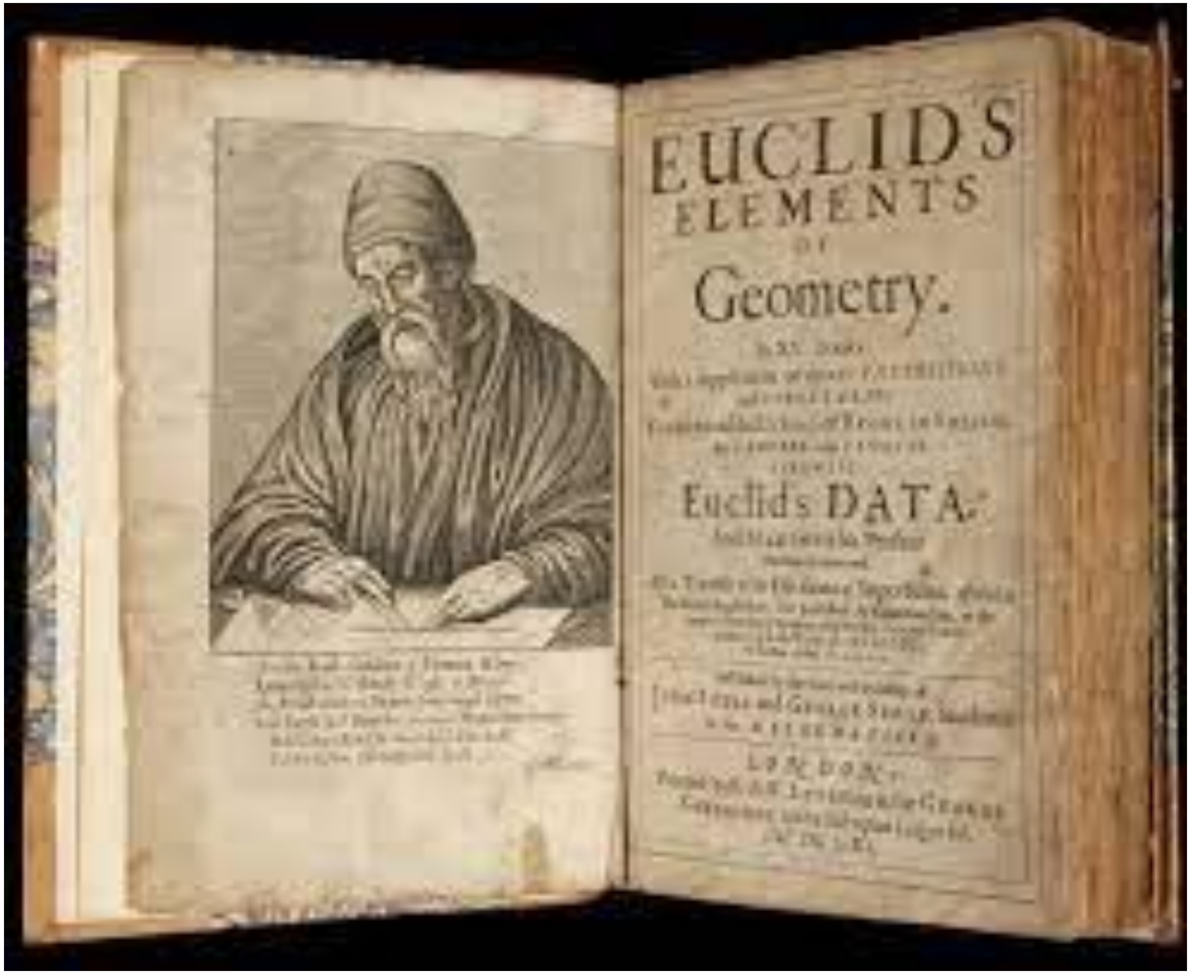
Pythagoras is often considered the first true mathematician.

The Pythagoreans' believed "All is Number," meaning that everything in the universe depended on numbers. They were also the first to teach that the Earth is a Sphere revolving around the sun.

The terms Mathematics (that which is learnt) appears

Themistokleia 6th century BC

Euclid 325 BC- 270 BC



Father of Geometry (Measuring the Earth).
Axiom, Lemma, Theorem, Proof.

Aristototelian-Ptolemaic cosmological model

Schema huius præmissæ diuisionis Sphærarum.

Autolycos, Eudoxos, Callipos, Apollonios, Conon, Arhimedes, Hipparchos, Sosigenes, Ptolemy

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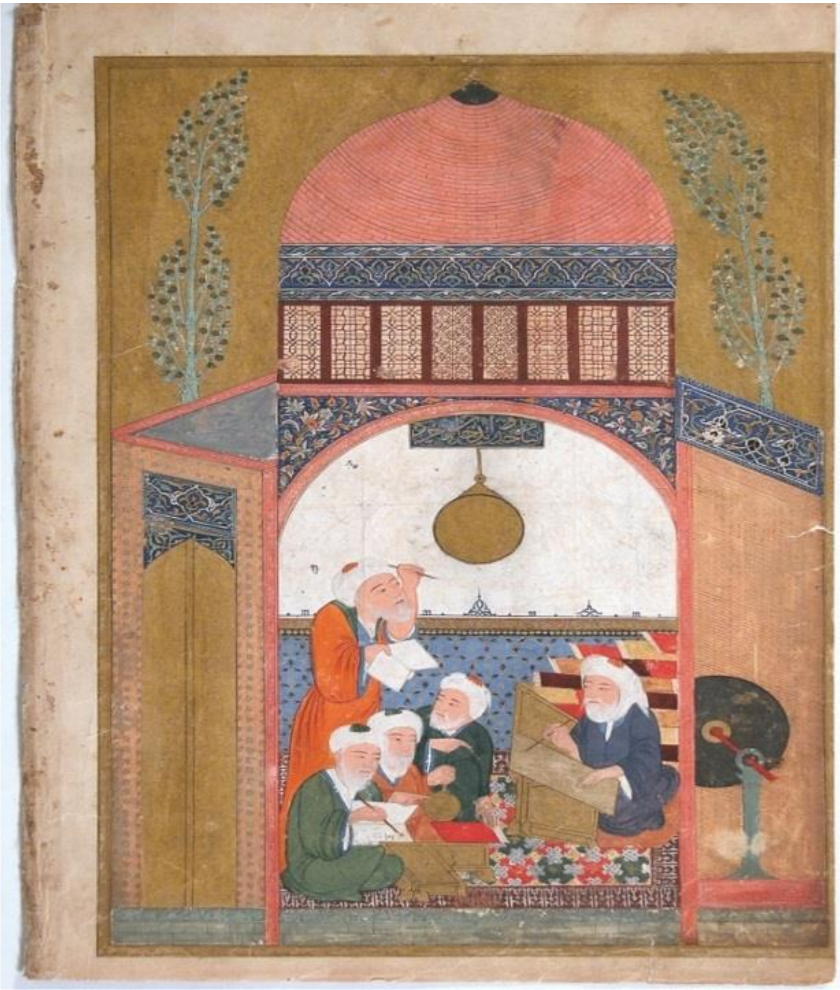
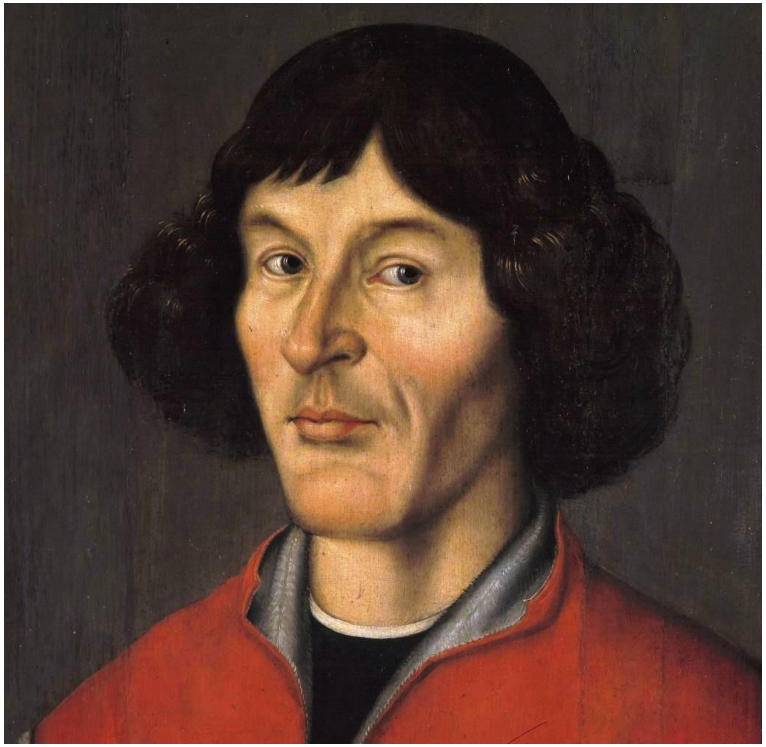
Maragha Observations

- Observations in Maragha in 11th century, started putting into doubt Earth's non-motion, however not geocentrism.



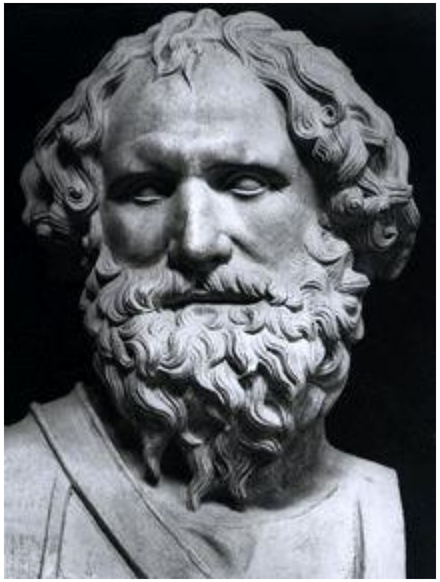
Copernicus - 1500

- Heliocentrism, (Aristarchus)



NICOLAI
COPERNICI TO-
RINENSIS DE REVOLVTIONI-
bus orbium cœlestium,
Libri VI.

BASILEAE, EX OFFICINA



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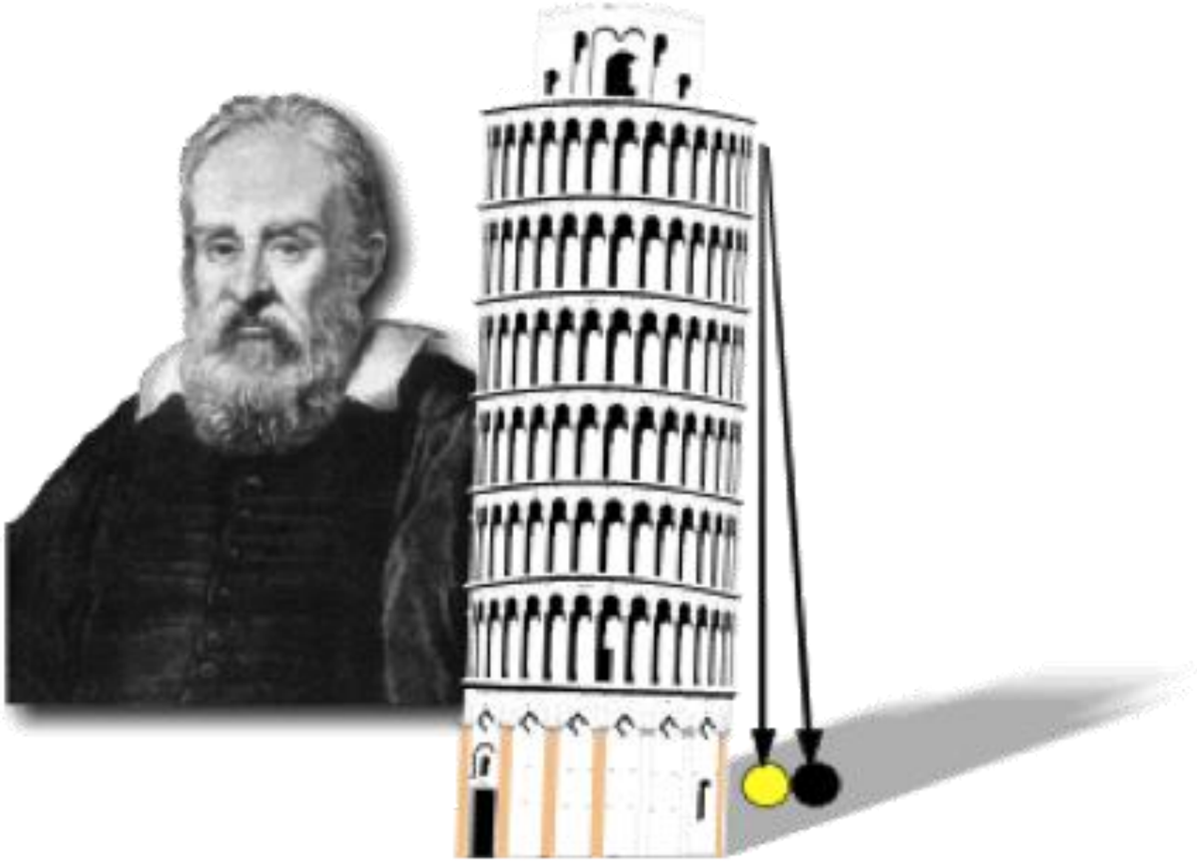
Brahe, Kepler- 1600

- Heliocentrism, elliptical Orbits

Galileo - 1600

- Bodies fall with the same speed, independently from their weight.



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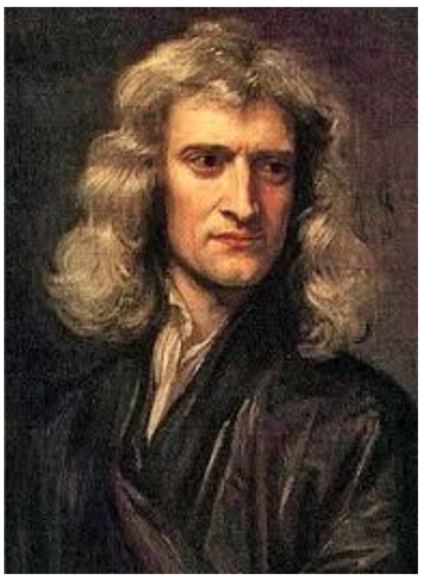
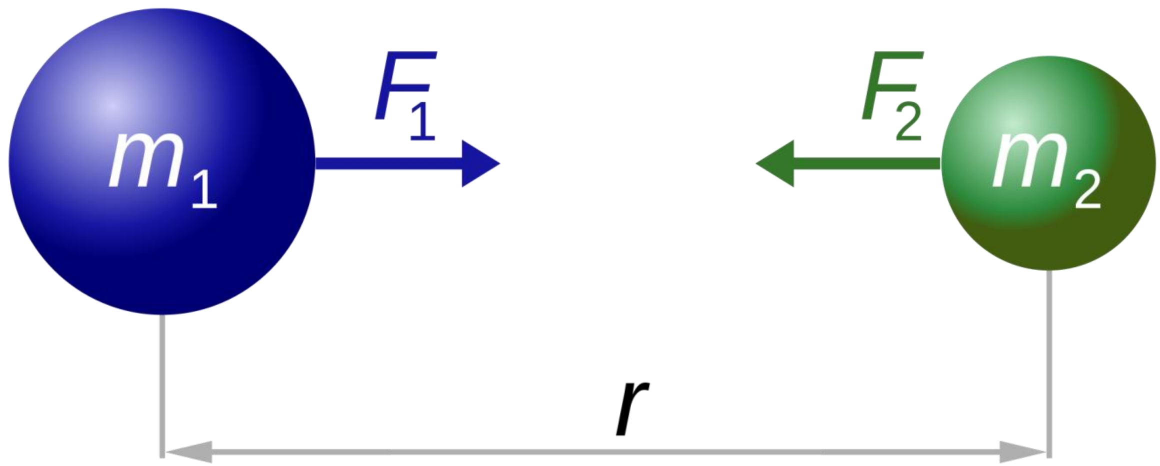
Newton - 1700

- Law of Universal Gravitation:

  All bodies (either apples or planets) attract mutually.

  First time that gravity is related to astronomy





$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

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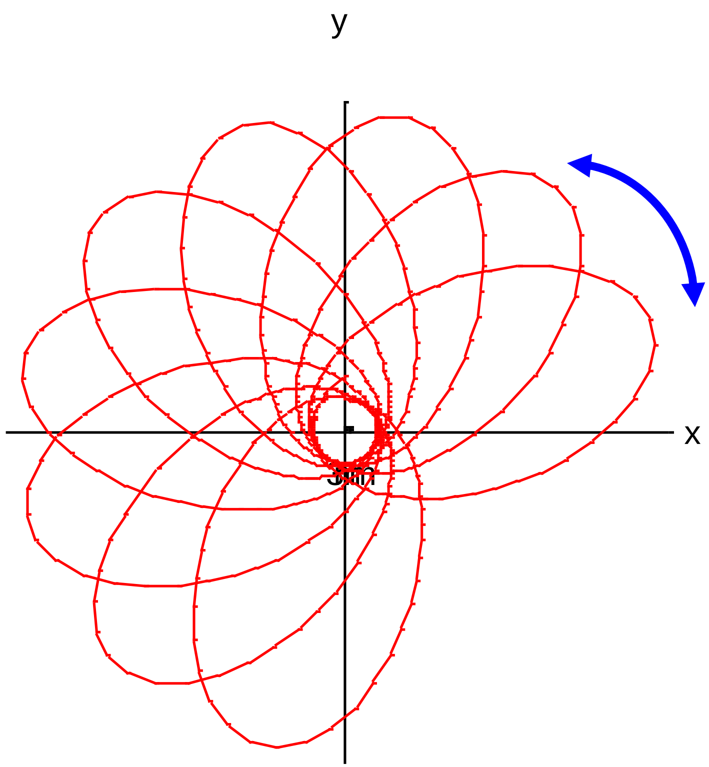
Mercury periliheimum - 1859

- *The true orbits of planets, even if seen from the SUN are not ellipses. They are rather curves of this type:*

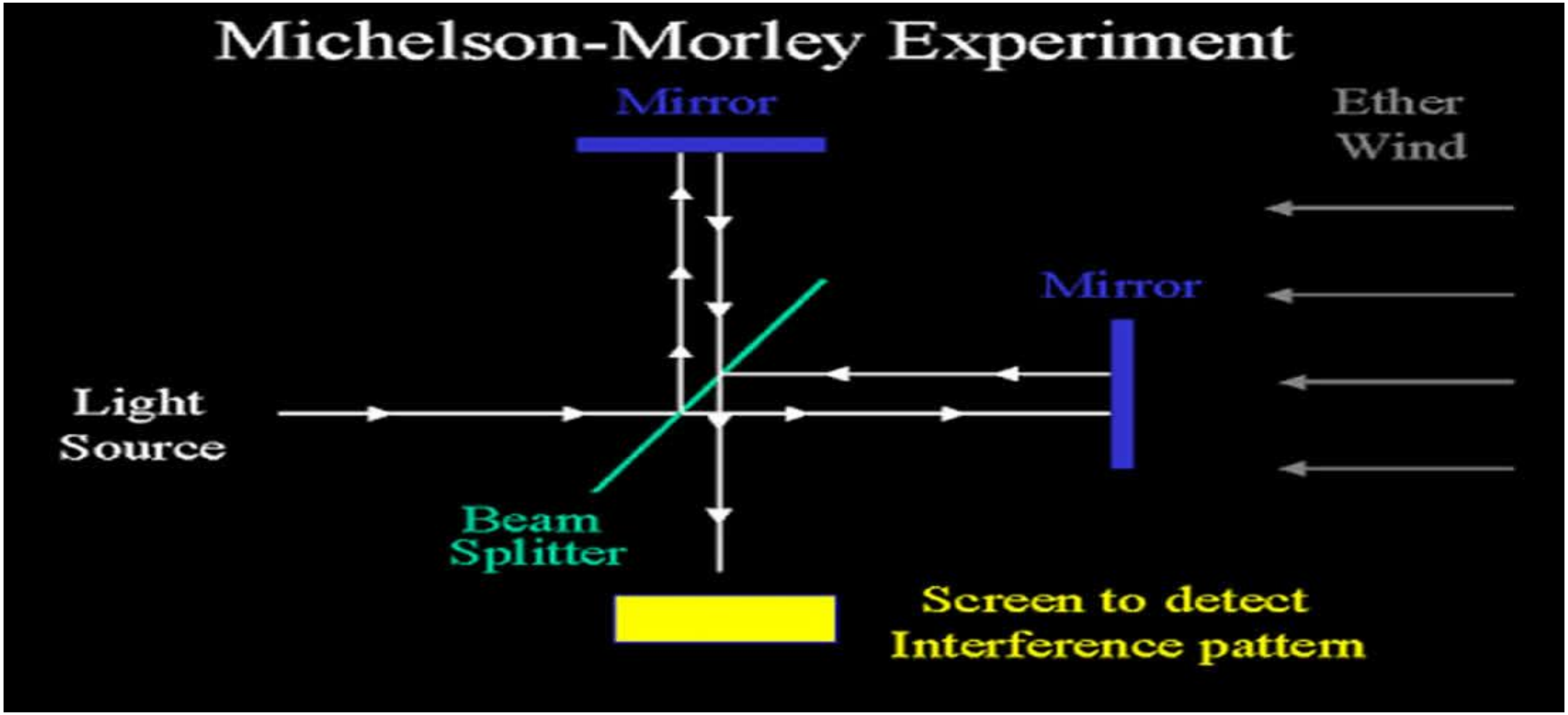


This angle is the perihelion advance, predicted by G.R.

For the planet Mercury it is

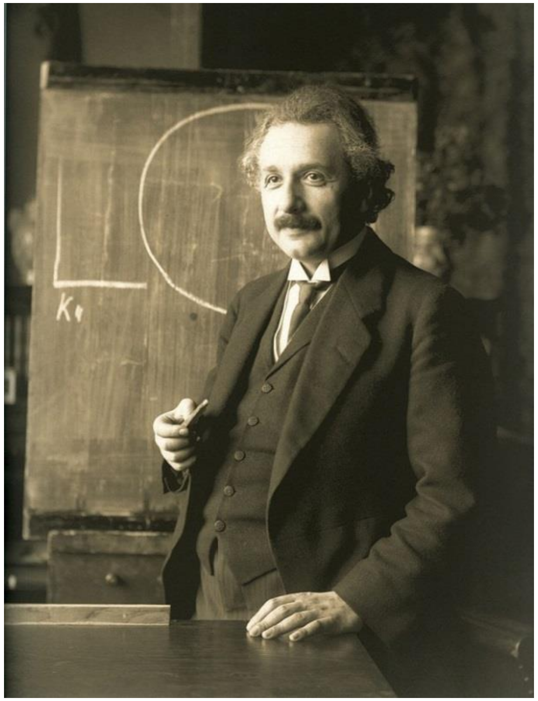
$$\Delta\varphi = 43'' \textit{ of arc per century}$$

Michelson–Morley experiment - 1887



General Relativity

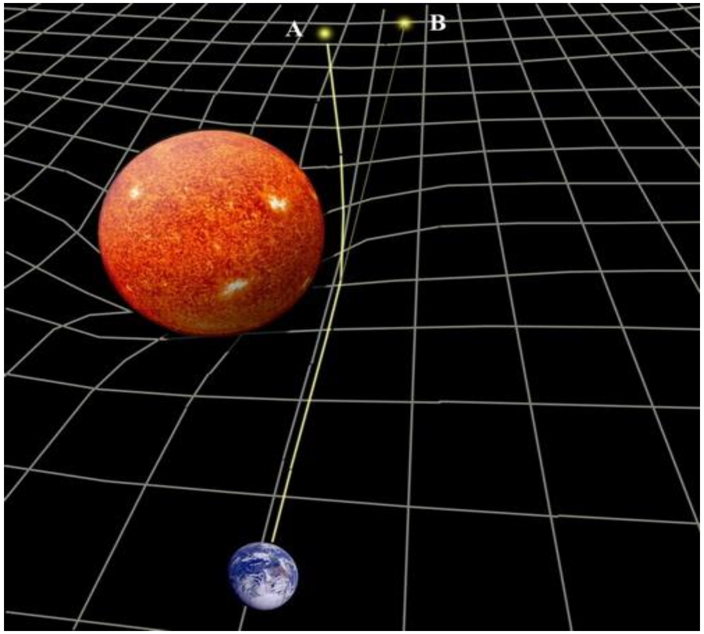
- The last theory from one scientist, the first collective theory



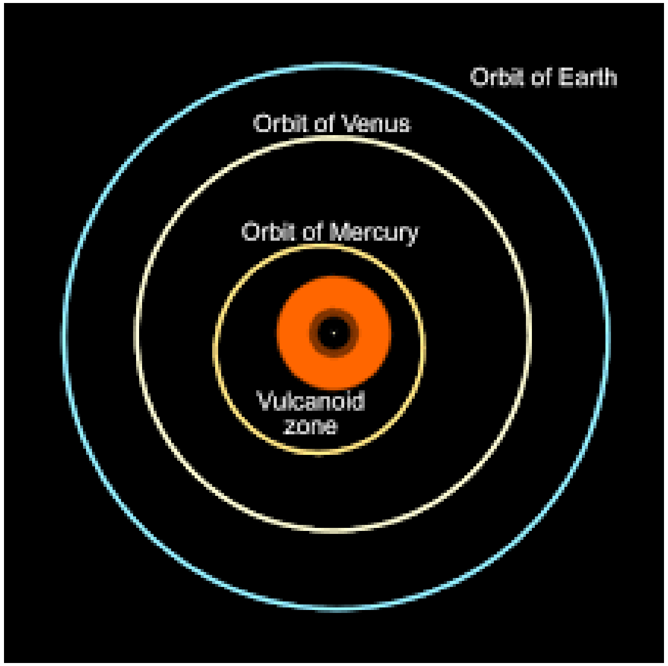
Gravity: General Relativity

- Matter tells spacetime how to curve

  Curved spacetime tells matter how to move



60 years of tension

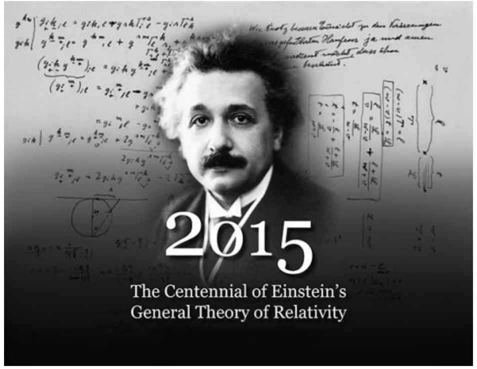


Modified Gravity before General Relativity

- Modifications to Newton's Law
- Inverse Cube Law.
- Extended Inverse-Square Law (Simon Newcomb -1880’s)
- Lord Kelvin - theory of everything (end of 19th century)
- Hendrik Lorentz: gravity on the basis of his ether theory and Maxwell's equations. (1900)
- Nordström's theory of gravitation (1912 and 1913)
- Einstein's scalar theory of gravity (1913)

General Relativity

- Einstein 1915: **General Relativity**:



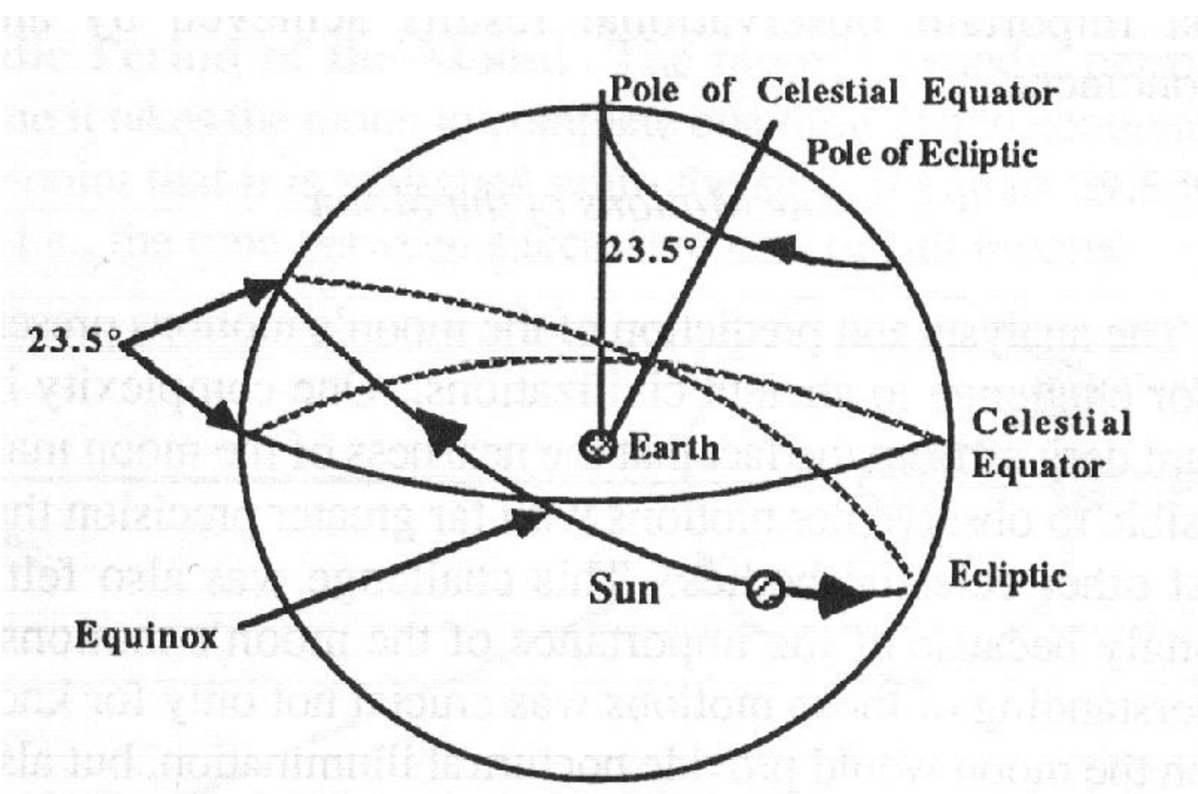
energy-momentum source of spacetime Curvature

$$\boxed{S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R-2\Lambda] + \int d^4x\, L_m(g_{\mu\nu},\psi)}$$

$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 8\pi G\, T_{\mu\nu}$$

with $\quad T^{\mu\nu} \equiv \dfrac{2}{\sqrt{-g}} \dfrac{\delta L_m}{\delta g_{\mu\nu}}$

The issue of reference frames and observers



Who is at motion? The Sun or the Earth?

A famous question with a lot of history behind it

Since oldest antiquity the humans have looked at the sky and at the motion of the Sun, the Moon and the Planets. Obviously they always did it from their reference frame, namely from the EARTH, which is not at rest, neither in rectilinear motion with constant velocity!

Classical Physics is founded.......

- on circular reasoning
- We have fundamental laws of Nature that apply only in special reference frames, **the inertial ones**
- How are the inertial frames defined?
- **As those where the fundamental laws of Nature apply**

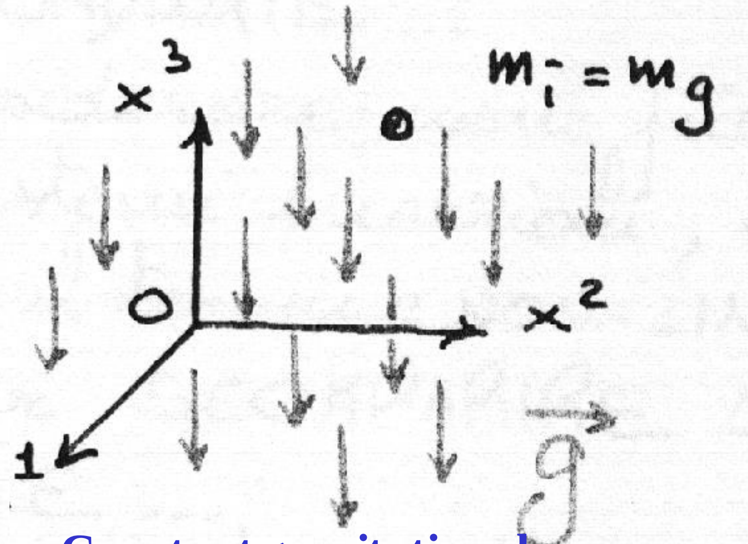
The idea of General Covariance

- It would be better if Natural Laws were formulated the same in whatever reference frame
- Whether we rotate with respect to distant galaxies or they rotate should not matter for the form of the Laws of Nature
- To agree with this idea we have to cast Laws of Nature into the language of geometry....

Equivalence Principle: a first approach

$$m_i \frac{d^2\vec{x}}{dt^2} = m_g \vec{g}$$

Newton's Law



Inertial and gravitational masses are equal

$$m_i = m_g$$

Constant gravitational field

Accelerated frame

$$\vec{y} = \vec{x} - \frac{1}{2}\vec{g}t^2$$

$$m_i \frac{d^2\vec{x}}{dt^2} = m_g \vec{g} \xrightarrow[y=y(x)]{} \frac{d^2 y}{dt^2} = 0$$

Gravity has been Locally suppressed

This is the Elevator Gedanken Experiment of Einstein

There is no way to decide whether we are in an accelerated frame or immersed in a locally constant gravitational field

The word *local* is crucial in this context!!

G.R. model of the physical world

Physics	*Geometry*
• The **when** and the **where** of any physical physical phenomenon constitute an **event.** • The set of all events is a continuous space, named **space-time** • **Gravitational phenomena** are manifestations of the **geometry** of space—time • **Point-like particles** move in space—time following special world-lines that are **“straight”** • The **laws of physics** are the same for **all observers**	• An **event** is a **point** in a topological space • **Space-time** is a **differentiable manifold** ***M*** • **The gravitational field** is a **metric** **g** on ***M*** • **Straight lines** are **geodesics** • **Field equations** are **generally covariant** under diffeomorphisms

Hence the mathematical model of space time is a pair:

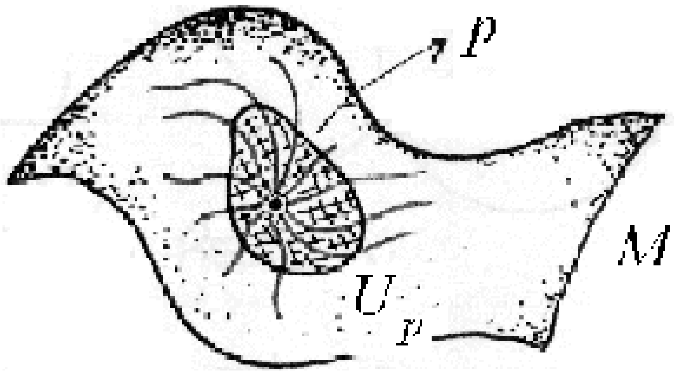
$$\left(M, g\right)$$

Differentiable Manifold

Metric

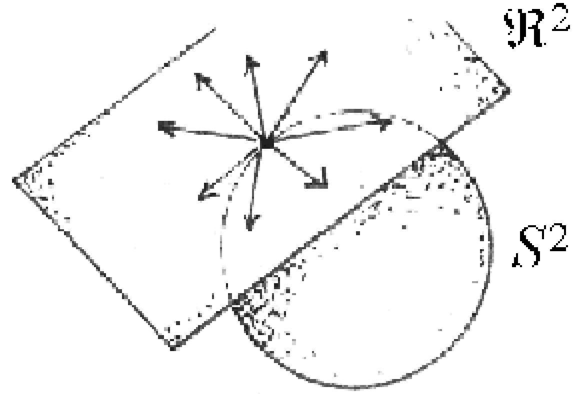
We need to review these two fundamental concepts

Tangent spaces and vector fields



In a neighborhood U_p of each point $p \in \mathcal{M}$

we consider the curves that go through p



The tangent space in a generic point of an $\mathbb{S}^2$ sphere

$$\boxed{\forall p \in \mathcal{M} \; : \; p \mapsto T_p\mathcal{M} \quad \dim T_p\mathcal{M} = m}$$

$T_p\mathcal{M}$ parametrizes the possible directions in which a curve starting at p can depart.

$$\vec{t}_p = c^\mu \frac{\vec{\partial}}{\partial x^\mu} \ni T_p\mathcal{M}$$

A tangent vector is a 1st order differential operator

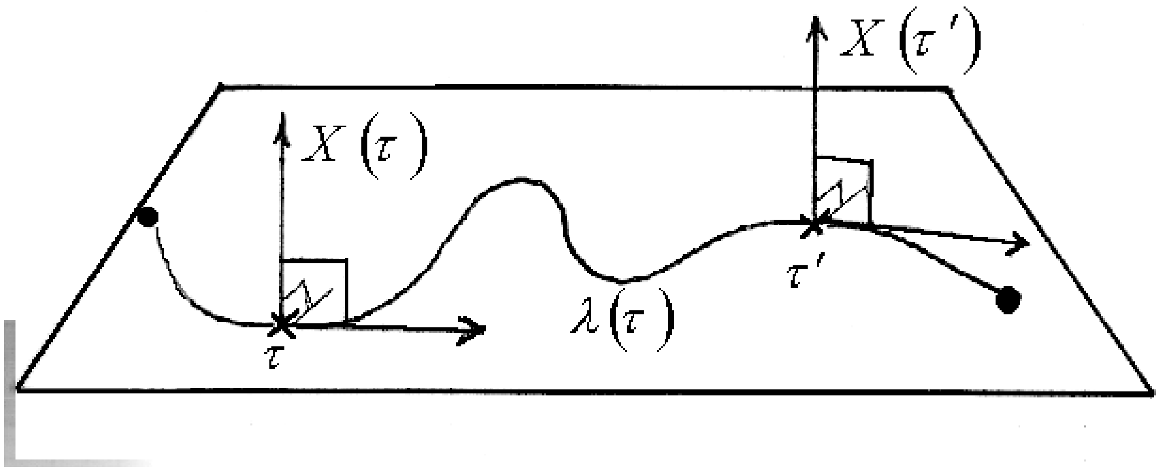
Under change of local coordinates

$$x^\mu = x^\mu(y) \quad ; \quad y^\alpha = y^\alpha(x)$$

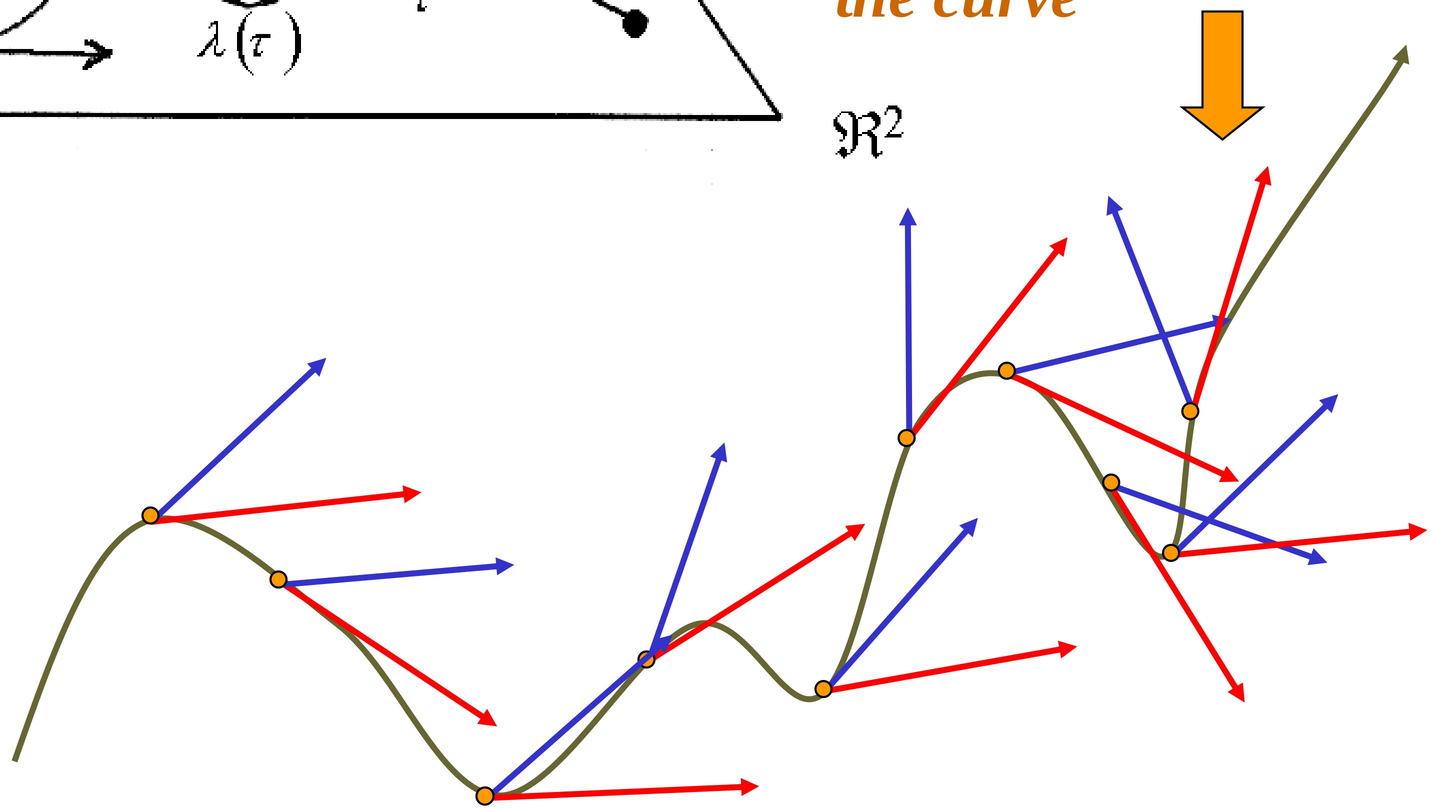
$$\vec{t}_p = c^\mu \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) \frac{\vec{\partial}}{\partial y^\alpha} = c^\alpha \frac{\vec{\partial}}{\partial y^\alpha}$$

$$c^\alpha \equiv c^\mu \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)$$

Parallel Transport

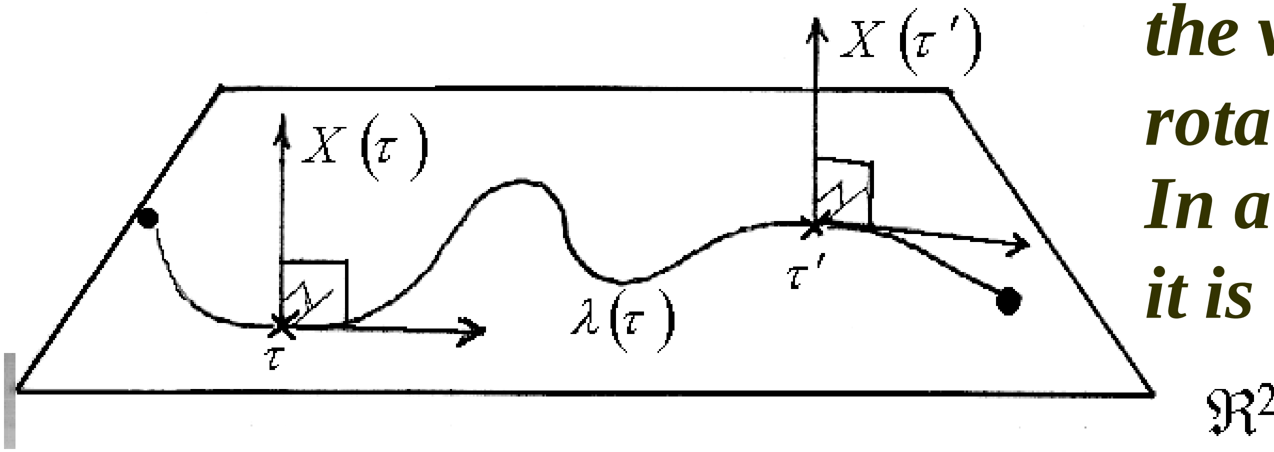


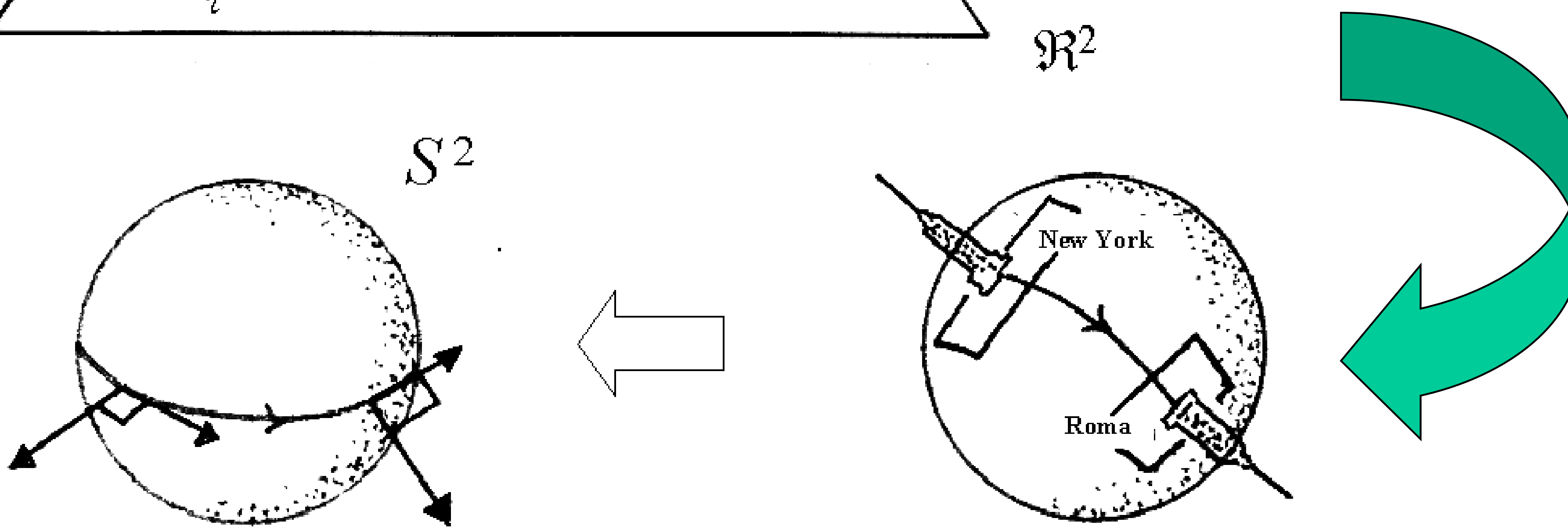
A vector field is parallel transported along a curve, when it mantains a constant angle with the tangent vector to the curve



The difference between flat and curved manifolds

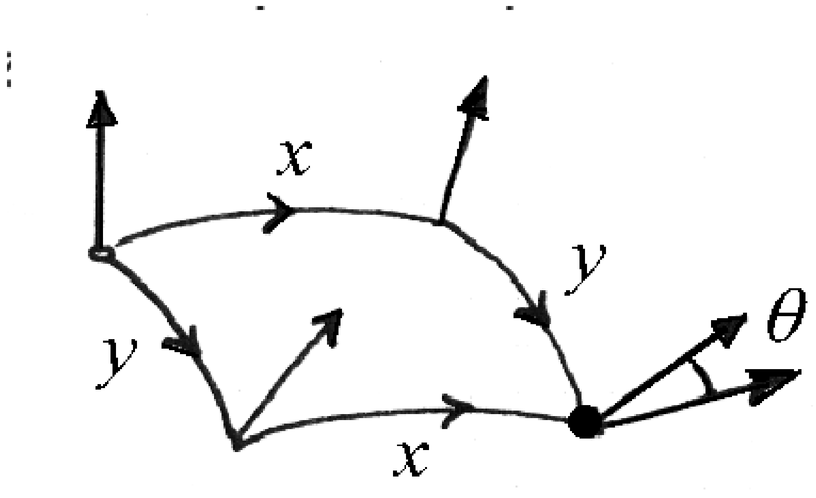
In a flat manifold, while transported, the vector is not rotated.
In a curved manifold it is rotated:





To see the real effect of curvature we must consider.....

Parallel transport along LOOPS



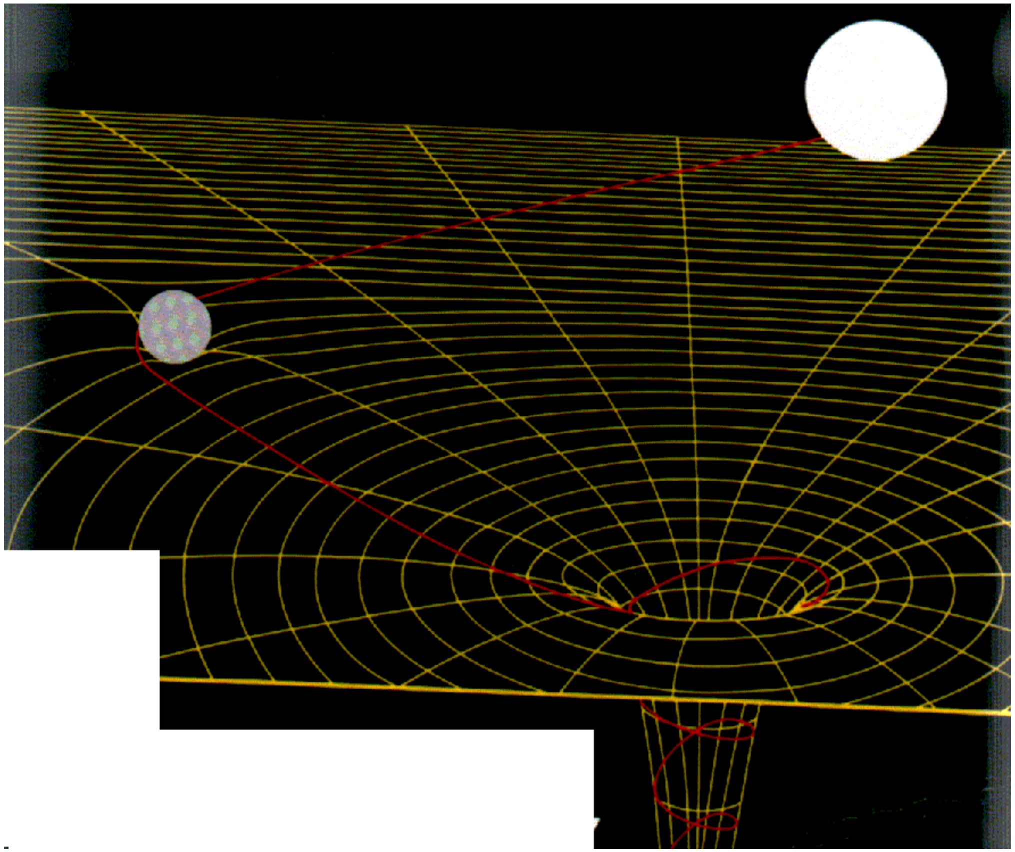
After transport along a loop, the vector does not come back to the original position but it is rotated of some angle.

What do particles do in a gravitational field?

Answer: They just go straight as in empty space!!!!

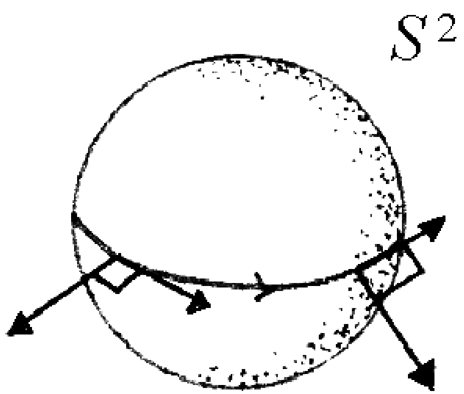
It is the concept of straight line that is modified by the presence of gravity!!!!

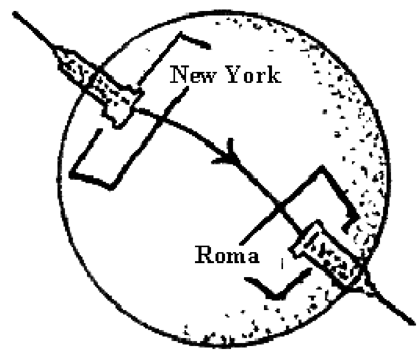
The metaphor of **Eddington's sheet** summarizes General Relativity.
In curved space *straight lines* are different from straight lines in flat space!!
The *red line* followed by the ball falling in the throat is *a straight line* (geodesics). On the other hand ***space-time is bended*** under the **weight of matter** moving inside it!



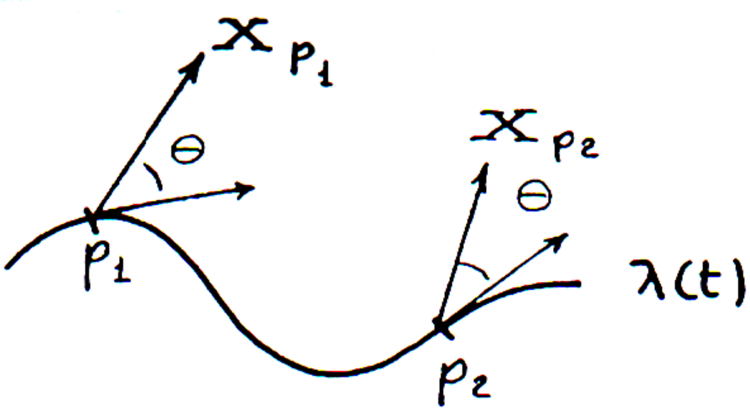
What are the straight lines

They are the geodesics, curves that do not change length under small deformations. ***These are the curves along which we have parallel transported our vectors***





On a sphere geodesics are maximal circles



In the parallel transport the angle with the tangent vector remains fixed. On geodesics the tangent vector is transported parallel to itself.

Let us now review the general case

1 Geodesics

The general equation for the geodesics is obtained from the variational principle

$$\delta \mathcal{L} = 0 \quad \text{where} \quad \mathcal{L} = -g_{\mu\nu}(x)\, \dot{x}^\mu\, \dot{x}^\nu$$

having set the proper time (or distance):

$$\int d\tau = \int \sqrt{-g_{\mu\nu}(x)\, \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$$

and we obtain

$$\boxed{0 = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\,\frac{dx^\sigma}{d\lambda}}$$

Christoffel symbols

=

Levi Civita connection

Variational Principle for Free Test Particle Motion

The world line of a free test particle between two timelike separated points extremizes the proper time between them.

TABLE 8.1 Extremal Proper time $\delta \int d\tau = 0$ and Equations of Motion

	Variational Principle	Equation of Motion
Particle in flat spacetime	$\delta \int (-\eta_{\alpha\beta} dx^\alpha dx^\beta)^{1/2} = 0$	$\frac{d^2x^\alpha}{d\tau^2} = 0$
Geometric Newtonian	$\delta \int \left[(1 + 2\Phi/c^2)(cdt)^2 - (1 - 2\Phi/c^2)(dx^2 + dy^2 + dz^2)\right]^{1/2} = 0$	$\frac{d^2x^i}{dt^2} = -\frac{\partial\Phi}{\partial x^i}$
	(to leading order in$1/c^2$)	(to leading order in $1/c^2$)
General metric	$\delta \int (-g_{\alpha\beta}\, dx^\alpha dx^\beta)^{1/2} = 0$	$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$

The procedure for finding the equations for timelike geodesics in spacetime is a straightforward generalization of Example 8.1. The proper time along a timelike world line between two points A and B in spacetime is, from (7.19),

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left[-g_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta\right]^{1/2}. \tag{8.7}$$

The world line can be described parametrically by giving the four coordinates x^α as a function of a parameter σ that varies between $\sigma = 0$ at endpoint A and $\sigma = 1$ at endpoint B. The proper time between A and B is then

$$\tau_{AB} = \int_0^1 d\sigma \left(-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right)^{1/2}. \tag{8.8}$$

The world lines that extremize the proper time between A and B are those that satisfy Lagrange's equations,

$$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\alpha/d\sigma)}\right) + \frac{\partial L}{\partial x^\alpha} = 0, \tag{8.9}$$

for the Lagrangian

$$L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha\right) = \left(-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right)^{1/2}. \tag{8.10}$$

4.2 Geodesic equation from variational principle

Consider motion of particle along a path $x^a(\tau)$. We will perform a variation on this path between two points P and Q. The action is simply

$$S = \int \mathrm{d}\tau. \tag{4.27}$$

In order to perform the variation, it is useful to introduce an arbitrary auxiliary parameter s. Here $\mathrm{d}s$ is displacement on spacetime. We have

$$\mathrm{d}\tau = \Big(- g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}s} \frac{\mathrm{d}x^b}{\mathrm{d}s} \Big)^{1/2} \mathrm{d}s \tag{4.28}$$

We vary the path by using standard procedure:

$$\begin{aligned} \delta S &= \delta \int \mathrm{d}\tau \\ &= \int \delta \Big(- g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}s} \frac{\mathrm{d}x^b}{\mathrm{d}s} \Big)^{1/2} \mathrm{d}s \\ &= \frac{1}{2} \int \mathrm{d}s \Big(- g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}s} \frac{\mathrm{d}x^b}{\mathrm{d}s} \Big)^{-1/2} \Big[- \delta g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}s} \frac{\mathrm{d}x^b}{\mathrm{d}s} - 2 g_{ab} \frac{\mathrm{d}\delta x^a}{\mathrm{d}s} \frac{\mathrm{d}x^b}{\mathrm{d}s} \Big]. \end{aligned}$$

Considering the last term,

$$-2g_{ab}\frac{\mathrm{d}\delta x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} \;=\; \frac{\mathrm{d}}{\mathrm{d}s}\Big[-2g_{ab}\delta x^a\frac{\mathrm{d}x^b}{\mathrm{d}s}\Big]+2\frac{\mathrm{d}g_{ab}}{\mathrm{d}s}\delta x^a\frac{\mathrm{d}x^b}{\mathrm{d}s}+2g_{ab}\delta x^a\frac{\mathrm{d}^2x^b}{\mathrm{d}s^2}.$$

The two points, P and Q are fixed. We can set first term of above equation to zero. Therefore we obtain

$$\begin{aligned}
\delta S &= \frac{1}{2}\int \mathrm{d}s\Big(-g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\Big)^{-1/2}\Big[-\delta g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}+2\frac{\mathrm{d}g_{ab}}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\delta x^a+2g_{ab}\frac{\mathrm{d}^2x^b}{\mathrm{d}s^2}\delta x^a\Big]\\
&= \frac{1}{2}\int \mathrm{d}\tau\frac{\mathrm{d}s^2}{\mathrm{d}\tau^2}\Big[-\delta g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}+2\frac{\mathrm{d}g_{ab}}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\delta x^a+2g_{ab}\frac{\mathrm{d}^2x^b}{\mathrm{d}s^2}\delta x^a\Big]\\
&= \frac{1}{2}\int \mathrm{d}\tau\Big[-\delta g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}+2\frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\delta x^a+2g_{ab}\frac{\mathrm{d}^2x^b}{\mathrm{d}\tau^2}\delta x^a\Big].
\end{aligned}$$

By using chain rule we get

$$\begin{aligned}
\delta S &= \frac{1}{2}\int \mathrm{d}\tau\Big[-\frac{\partial g_{ab}}{\partial x^c}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\delta x^c+2\frac{\partial g_{ab}}{\partial x^c}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\delta x^a+2g_{ab}\frac{\mathrm{d}^2x^b}{\mathrm{d}\tau^2}\delta x^a\Big]\\
&= \frac{1}{2}\int \mathrm{d}\tau\Big[-\partial_b g_{ac}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\delta x^b+\partial_c g_{ba}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\delta x^b+\partial_a g_{bc}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\delta x^b+2g_{ba}\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2}\delta x^b\Big]\\
&= \int \mathrm{d}\tau\delta x^b\Big[\frac{1}{2}\Big(-\partial_b g_{ac}+\partial_c g_{ba}+\partial_a g_{bc}\Big)\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}+g_{ba}\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2}\Big].
\end{aligned}$$

We set the variation of action to zero, $\delta S = 0$ and multiply by g^{db},

$$\frac{1}{2}g^{db}\Big(-\partial_b g_{ac} + \partial_c g_{ba} + \partial_a g_{bc}\Big)\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} + \delta^d_a\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2} = 0$$

We therefore recover geodesic equation:

$$\frac{\mathrm{d}^2x^d}{\mathrm{d}\tau^2} + \Gamma^d_{ac}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} \quad = \quad 0$$

The Christoffel symbols are:

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left\{ \partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma} \right\}$$

Where from do they emerge and what is their meaning?

ANSWER:

They are the coefficients of an **affine connection**, namely the proper mathematical concept underlying the concept of parallel transport.

Let us review the **concept of connection**

It suffices that the field equations be of the form:

$$G_{ab} = 4\pi G\, T_{ab} \Longrightarrow D^a T_{ab} = 0$$

•Source of gravity in Newton's theory is the mass

•In Relativity mass and energy are interchangeable. Hence Energy must be the source of gravity.

•Energy is not a scalar, it is the 0th component of 4-momentum. Hence 4—momentum must be the source of gravity

•The current of 4—momentum is the stress energy tensor. It has just so many components as the metric!!

•Einstein tensor is the unique tensor, quadratic in derivatives of the metric that couples to stress-energy tensor consistently

$$
\begin{aligned}
\delta S_{\mathrm{EH}} &= \delta \int \sqrt{-g} R \, \mathrm{d}^4 x \\
&= \int \mathrm{d}^4 x \delta\Big(\sqrt{-g} g^{ab} R_{ab}\Big) \\
&= \int \mathrm{d}^4 x \sqrt{-g} g^{ab} \delta R_{ab} + \int \mathrm{d}^4 x \sqrt{-g} R_{ab} \delta g^{ab} + \int \mathrm{d}^4 x R \delta \sqrt{-g}.
\end{aligned}
$$

Now we have three terms of variation that

$$
\delta S_{\mathrm{EH}} = \delta S_{\mathrm{EH}(1)} + \delta S_{\mathrm{EH}(2)} + \delta S_{\mathrm{EH}(3)} \tag{4.3}
$$

The variation of first term is

$$
\delta S_{\mathrm{EH}(1)} = \int \mathrm{d}^4 x \sqrt{-g} g^{ab} \delta R_{ab}. \tag{4.4}
$$

Considering the variation of Ricci tensor,

$$\begin{aligned} R_{ab} = R^c{}_{acb} &= \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{cd}\Gamma^d_{ba} - \Gamma^c_{bd}\Gamma^d_{ac} \qquad (4.5)\\ \delta R_{ab} &= \partial_c \delta\Gamma^c_{ab} - \partial_b \delta\Gamma^c_{ac} + \Gamma^d_{ba}\delta\Gamma^c_{cd} + \Gamma^c_{cd}\delta\Gamma^d_{ba} - \Gamma^d_{ac}\delta\Gamma^c_{bd} - \Gamma^c_{bd}\delta\Gamma^d_{ac} \\ &= \left(\partial_c \delta\Gamma^c_{ab} + \Gamma^c_{cd}\delta\Gamma^d_{ba} - \Gamma^d_{ac}\delta\Gamma^c_{bd} - \Gamma^d_{bc}\delta\Gamma^c_{ad}\right) \\ &\quad -\left(\partial_b \delta\Gamma^c_{ac} + \Gamma^c_{bd}\delta\Gamma^d_{ac} - \Gamma^d_{ba}\delta\Gamma^c_{cd} - \Gamma^d_{bc}\delta\Gamma^c_{ad}\right) \end{aligned}$$

and the covariant derivative formula:

$$\nabla_c \delta\Gamma^c_{ab} = \partial_c \delta\Gamma^c_{ab} + \Gamma^c_{cd}\delta\Gamma^d_{ba} - \Gamma^d_{ac}\delta\Gamma^c_{bd} - \Gamma^d_{bc}\delta\Gamma^c_{ad} \qquad (4.6)$$

and also

$$\nabla_b \delta\Gamma^c_{ac} = \partial_b \delta\Gamma^c_{ac} + \Gamma^c_{bd}\delta\Gamma^d_{ac} - \Gamma^d_{ba}\delta\Gamma^c_{cd} - \Gamma^d_{bc}\delta\Gamma^c_{ad} \qquad (4.7)$$

we can conclude that

$$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}. \qquad (4.8)$$

$$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}. \tag{4.8}$$

Therefore equation (4.4) becomes

$$\begin{aligned}
\delta S_{\mathrm{EH}(1)} &= \int \mathrm{d}^4x\sqrt{-g}g^{ab}\Big(\nabla_c\delta\Gamma^c_{ab} - \nabla_b\delta\Gamma^c_{ac}\Big) \\
&= \int \mathrm{d}^4x\sqrt{-g}\Big[\nabla_c\Big(g^{ab}\delta\Gamma^c_{ab}\Big) - \delta\Gamma^c_{ab}\nabla_c g^{ab} - \nabla_b\Big(g^{ab}\nabla_b\delta\Gamma^c_{ac}\Big) + \delta\Gamma^c_{ac}\nabla_b g^{ab}\Big].
\end{aligned}$$

Remembering that the covariant derivative of metric is zero. Therefore we get

$$\begin{aligned}
\delta S_{\mathrm{EH}(1)} &= \int \mathrm{d}^4x\sqrt{-g}g^{ab}\Big(\nabla_c\delta\Gamma^c_{ab} - \nabla_b\delta\Gamma^c_{ac}\Big) \\
&= \int \mathrm{d}^4x\sqrt{-g}\Big[\nabla_c\Big(g^{ab}\delta\Gamma^c_{ab}\Big) - \nabla_b\Big(g^{ab}\delta\Gamma^c_{ac}\Big)\Big] \\
&= \int \mathrm{d}^4x\sqrt{-g}\,\nabla_c\Big[g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^b_{ab}\Big] \\
&= \int \mathrm{d}^4x\sqrt{-g}\,\nabla_c J^c
\end{aligned}$$

where we introduce

$$J^c = g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^b_{ab}. \tag{4.9}$$

If J^c is a vector field over a region $\mathcal{M}$ with boundary Σ. Stokes's theorem for the vector field is

$$\int_{\mathcal{M}} \mathrm{d}^4x \sqrt{|g|}\, \nabla_c J^c = \int_{\Sigma} \mathrm{d}^3x \sqrt{|h|} n_c J^c \tag{4.10}$$

where n_c is normal unit vector on hypersurface Σ. The normal unit vector n_c can be normalized by $n_a n^a = -1$. The tensor h_{ab} is induced metric associated with hypersurface defined by

$$h_{ab} = g_{ab} + n_a n_b. \tag{4.11}$$

Therefore the first term of action becomes

$$\delta S_{\mathrm{EH}(1)} = \int_{\Sigma} \mathrm{d}^3x \sqrt{|h|} n_c J^c = 0. \tag{4.12}$$

This equation is an integral with respect to the volume element of the covariant divergence of a vector. Using Stokes's theorem, this is equal to a boundary contribution at infinity which can be set to zero by vanishing of variation at infinity. Therefore this term contributes nothing to the total variation.

4.1.2 Variation of the metrics

Firstly we consider metric g_{ab}. Since the contravariant and covariant metrics are symmetric matrices then,

$$g_{ca}g^{ab} = \delta_c{}^b. \tag{4.13}$$

We now consider inverse of the metric:

$$g^{ab} = \frac{1}{g}(A^{ab})^{\mathrm{T}} = \frac{1}{g}A^{ba} \tag{4.14}$$

$$\frac{\partial g}{\partial g_{ab}} = A^{ab}. \tag{4.16}$$

Let us consider variation of determinant g:

$$\begin{aligned} \delta g &= \frac{\partial g}{\partial g_{ab}}\delta g_{ab} \\ &= A^{ab}\delta g_{ab} \\ &= g g^{ba}\delta g_{ab}. \end{aligned}$$

Remembering that g^{ab} is symmetric, we get

$$\delta g = g g^{ab}\delta g_{ab}. \tag{4.17}$$

Using relation obtained above, we get

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\,\delta g \\ &= \frac{1}{2}\frac{g}{\sqrt{-g}}\,g^{ab}\delta g_{ab}. \end{aligned} \tag{4.18}$$

We shall convert from δg_{ab} to δg^{ab} by considering

$$\begin{aligned} \delta \delta_a{}^d = \delta(g_{ac} g^{cd}) &= 0 \\ g^{cd} \delta g_{ac} + g_{ac} \delta g^{cd} &= 0 \\ g^{cd} \delta g_{ac} &= -g_{ac} \delta g^{cd}. \end{aligned}$$

Multiply both side of this equation by g_{bd} we therefore have

$$\begin{aligned} g_{bd} g^{cd} \delta g_{ac} &= -g_{bd} g_{ac} \delta g^{cd} \\ \delta_b^c \delta g_{ac} &= -g_{bd} g_{ac} \delta g^{cd} \\ \delta g_{ab} &= -g_{ac} g_{bd} \delta g^{dc}. \end{aligned} \tag{4.19}$$

Substituting this equation in to equation (4.18) we obtain

$$\begin{aligned} \delta \sqrt{-g} &= -\frac{1}{2} \sqrt{-g}\, g^{ab} g_{ac} g_{bd} \delta g^{dc} \\ &= -\frac{1}{2} \sqrt{-g}\, \delta_c^b g_{bd} \delta g^{dc} \\ &= -\frac{1}{2} \sqrt{-g}\, g_{cd} \delta g^{dc}. \end{aligned}$$

Renaming indices c to a and d to b, we get

$$\delta\sqrt{-g} \;=\; -\frac{1}{2}\sqrt{-g}\, g_{ab}\delta g^{ab}. \tag{4.20}$$

The variation of Einstein-Hilbert action becomes

$$\begin{aligned}\delta S_{\mathrm{EH}} &= \int \mathrm{d}^4x\sqrt{-g}R_{ab}\delta g^{ab} - \frac{1}{2}\int \mathrm{d}^4x R\sqrt{-g}\, g_{ab}\delta g^{ab} \\ &= \int \mathrm{d}^4x\sqrt{-g}\Big[R_{ab} - \frac{1}{2}g_{ab}R\Big]\delta g^{ab} \end{aligned} \tag{4.21}$$

The functional derivative of the action satisfies

$$\delta S = \int \sum_i \left(\frac{\delta S}{\delta \Phi^i}\delta\Phi^i\right)\mathrm{d}^4x \tag{4.22}$$

where $\{\Phi^i\}$ is a complete set of field varied. Stationary points are those for which $\delta S/\delta\Phi^i = 0$. We now obtain Einstein's equation in vacuum:

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{EH}}}{\delta g^{ab}} = R_{ab} - \frac{1}{2}g_{ab}R = 0. \tag{4.23}$$

$$S = \frac{1}{16\pi G} S_{\mathrm{EH}} + S_{\mathrm{M}} \tag{4.24}$$

where S_{M} is the action for matter. We normalize the gravitational action so that we get the right answer. Following the above equation we have

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{ab}} = \frac{1}{16\pi G} \left(R_{ab} - \frac{1}{2} g_{ab} R \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_{\mathrm{M}}}{\delta g^{ab}} = 0.$$

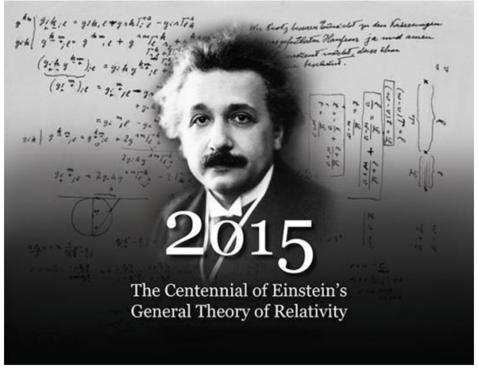
We now define the energy-momentum tensor as

$$T_{ab} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_{\mathrm{M}}}{\delta g^{ab}}. \tag{4.25}$$

This allows us to recover the complete Einstein's equation,

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab}. \tag{4.26}$$

General Relativity



Energy-momentum source of spacetime Curvature

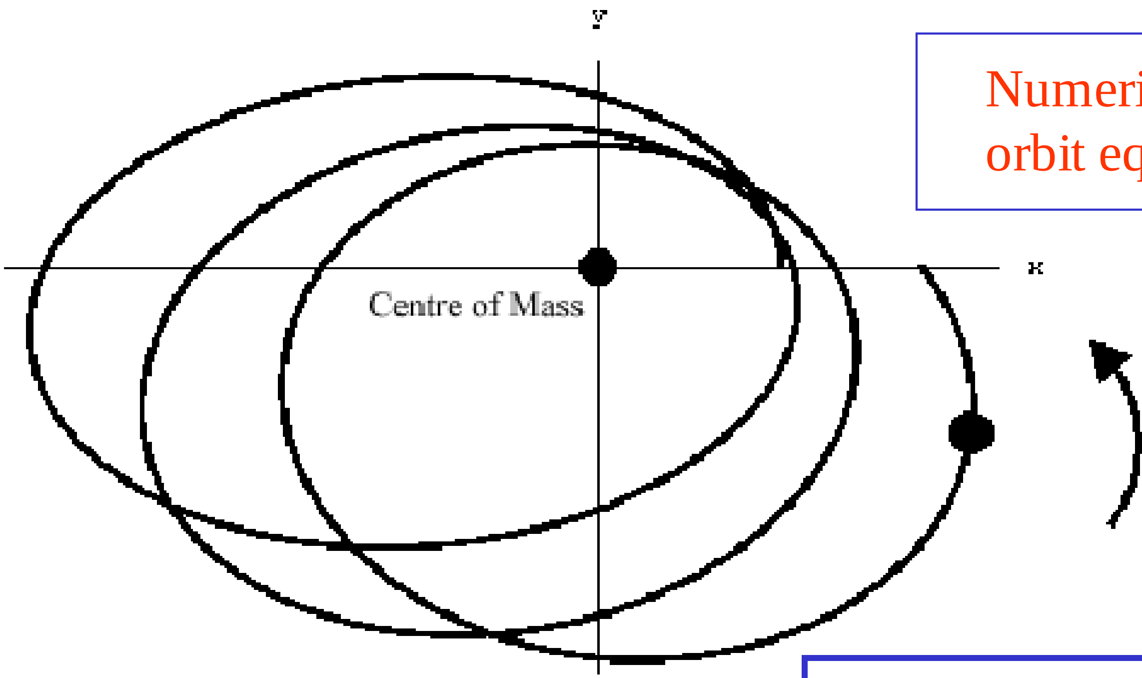
$$\boxed{S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R-2\Lambda\right]+\int d^4x\, L_m\left(g_{\mu\nu},\psi\right)}$$

$$\Rightarrow \quad R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda=8\pi G\,T_{\mu\nu}$$

with $$T^{\mu\nu}\equiv\frac{2}{\sqrt{-g}}\frac{\delta L_m}{\delta g_{\mu\nu}}$$

The effects: Periastron Advance

Figure 8: Schwarzschild geometry: orbit of a massive test particle with Keplerian parameters $a = 70m$ and $e = 0.7$ after 3 revolution

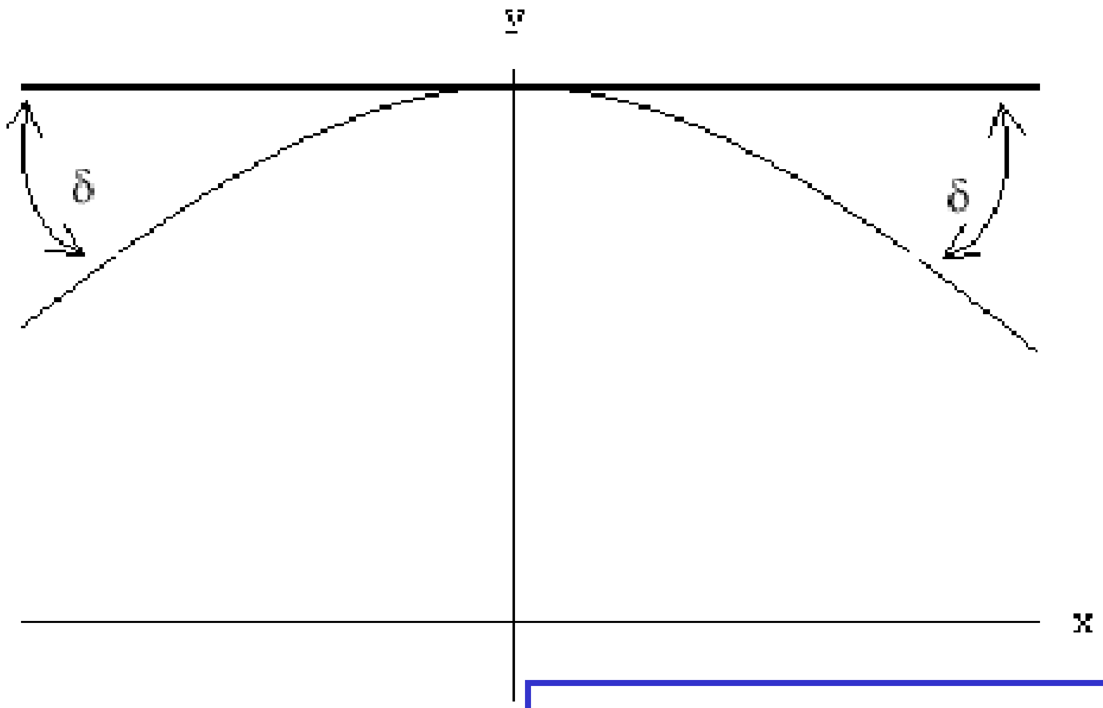


Numerical solution of orbit equation in G.R.

$$\Delta\phi = \frac{6\pi\, m}{a(1-e^2)}$$

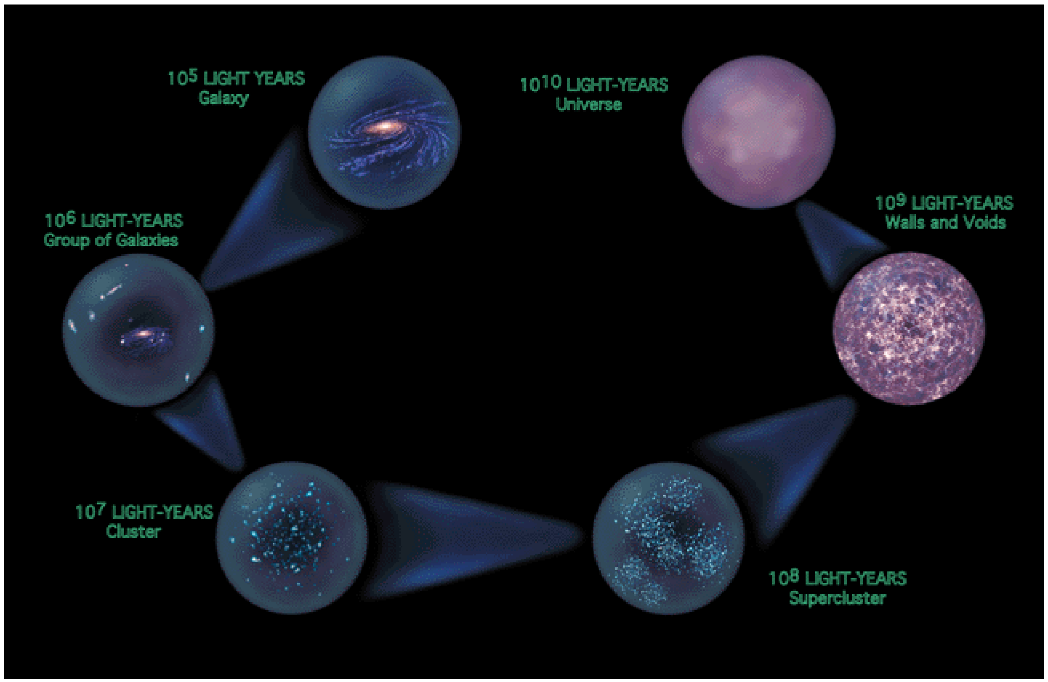
Bending of Light rays

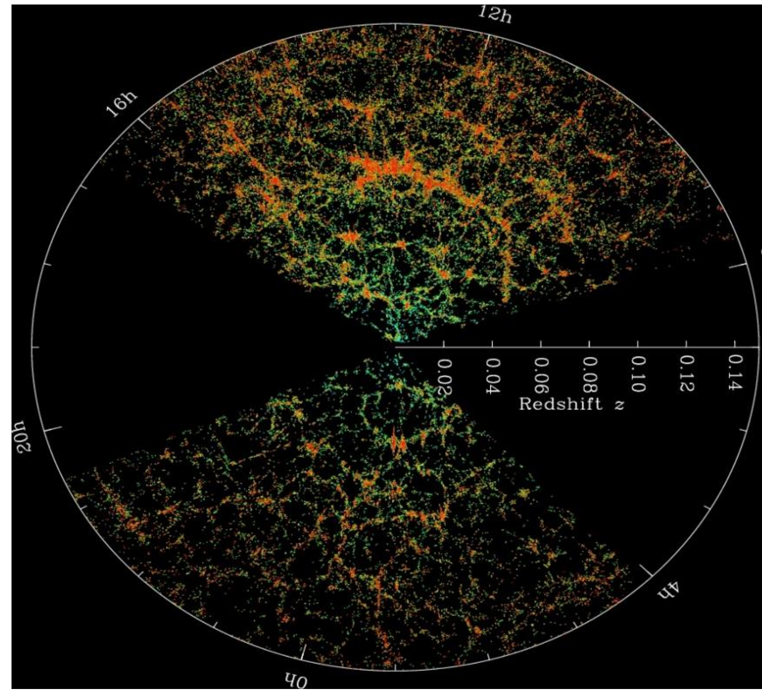
Figure 20: Comparison between the newtonian and postnewtonian perturbed trajectory of a photon impinging on a centre of mass with impact parameter $b = 30\,m$ and incidence angle $\phi_0 = 0$



$$\delta = 2\,\frac{m}{b} = \Delta\,\phi_{light} = 4\,\frac{M\,G}{c^2}\,\frac{1}{b}$$

Observations





- SDSS (Sloan Digital Sky Survey) 2004: ~ clusters "above and below the galactic plane" up to 1 Gpc

Observations

- As the scale we observe the Universe increases, it looks as homogeneous and isotropic.

- Cosmological Principle: "axiom" (indirect result)

  I) We know that earth is an isotropic observation point.

  II) An anisotropic system has up to one isotropic observation point.

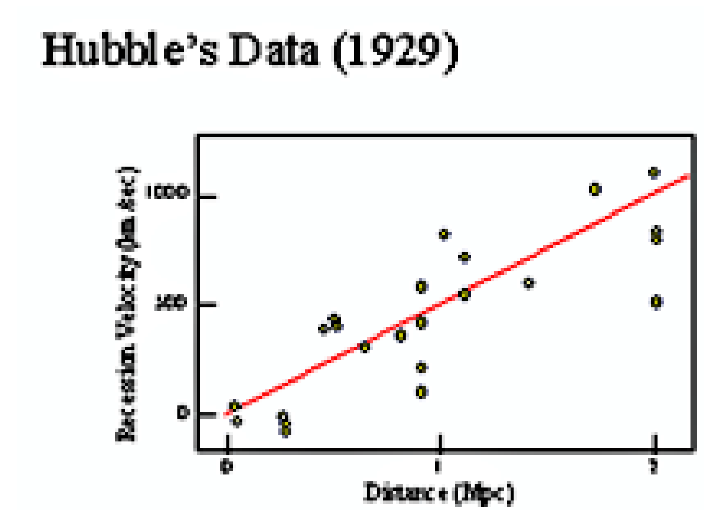
- Hence, either we lie in the only isotropic observation point in an anisotropic Universe, or all its points are isotropic observation points.

- Thus, the Universe is homogeneous and isotropic (isotropic and inhomogeneous is not possible)

Observations

- Hubble 1929: The Universe expands

Hubble excelled in every course at school (except spelling), but was better known for his athletic prowess. He was a star player in football, baseball, and basketball, and ran track in high school and at the University of Chicago, where he earned a Bachelor of Science in 1910.

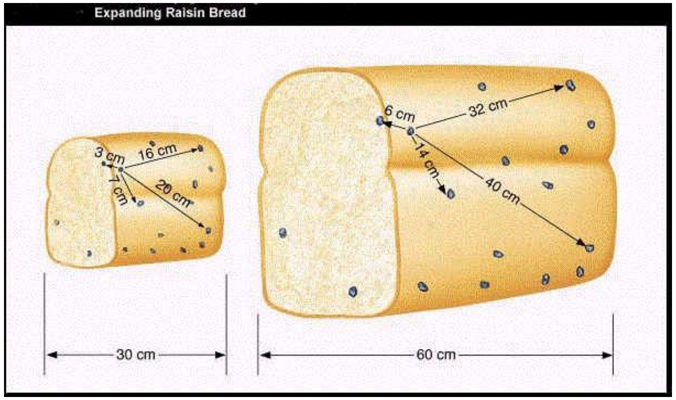


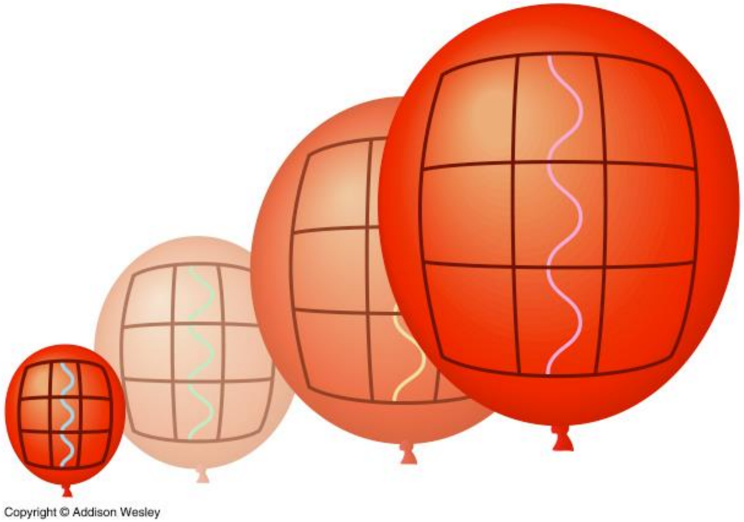
$$\boxed{v = H\ r}$$

$$H_0 \approx 70\ km\ s^{-1}\ Mpc^{-1}$$

Expansion

$$z = \frac{\Delta\lambda}{\lambda} = \sqrt{\frac{c+v}{c-v}} - 1 \quad \approx \frac{v}{c}$$





Observations

- Since the Universe expands it is reasonable that it originates from a “too tiny” and “too dense” “primordial atom” (Lemaitre 1927)

- Alpher, Bethe, Gamow (1948): The Universe begun to expand from a very high-density and high-temperature state towards less dense and hot states. Hoyle named the theory “The Big Bang Theory“.

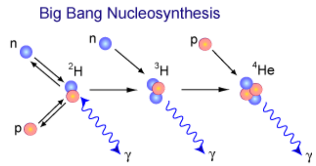
$$\boxed{t_U = \frac{r}{v} = \frac{r}{Hr} = \frac{1}{H} = \frac{1}{70}\left[\frac{Mpc}{km}\right] s \approx 14\ Gy}$$

- **Prediction I**: Nucleosynthesis has primordial origin, namely at first 3 minutes ($\sim 10^9$ K) (giving 25% Helium) and not in stars (1-4%)

  As observed.

Observations

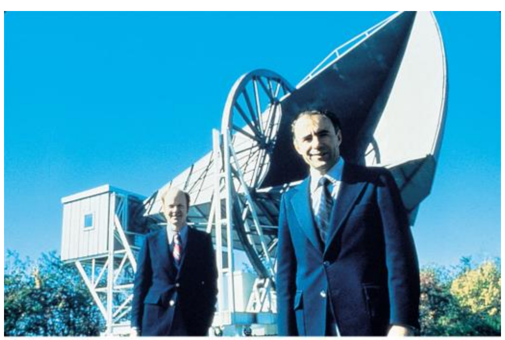
- **Prediction II**: The primordial Universe became full of high-energy photons

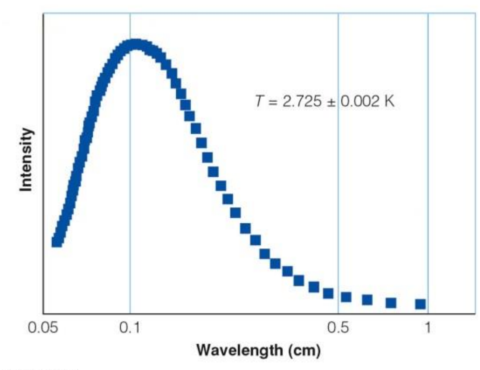


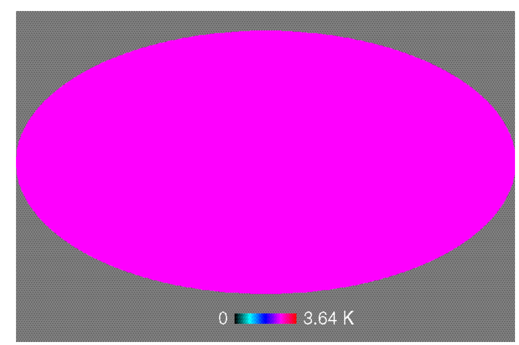
$$\lambda \approx 7 \cdot 10^{-12}\ cm,$$

380.000 years after (~3000K) they decouple from electrons (Recombination era). Black body radiation (today ~2.7 K)

- 1965 Penzias και Wilson







Theoretical arguments

- Big Bang Theory explained: Olbers paradox (1826) (why night sky is not bright), Ryle (1970) (Radio galaxies density increases with redshift), Element abundance, CMB, etc

- Theoretical Problems:
- I) Horizon problem: Why points at opposite directions have the same properties
- II) Flatness problem: Why the universe is today almost spatially flat Ω_k ~0.001. It must have started with $\sim 10^{-50}$!
- Monopole problem: They are not observed.

Inflation

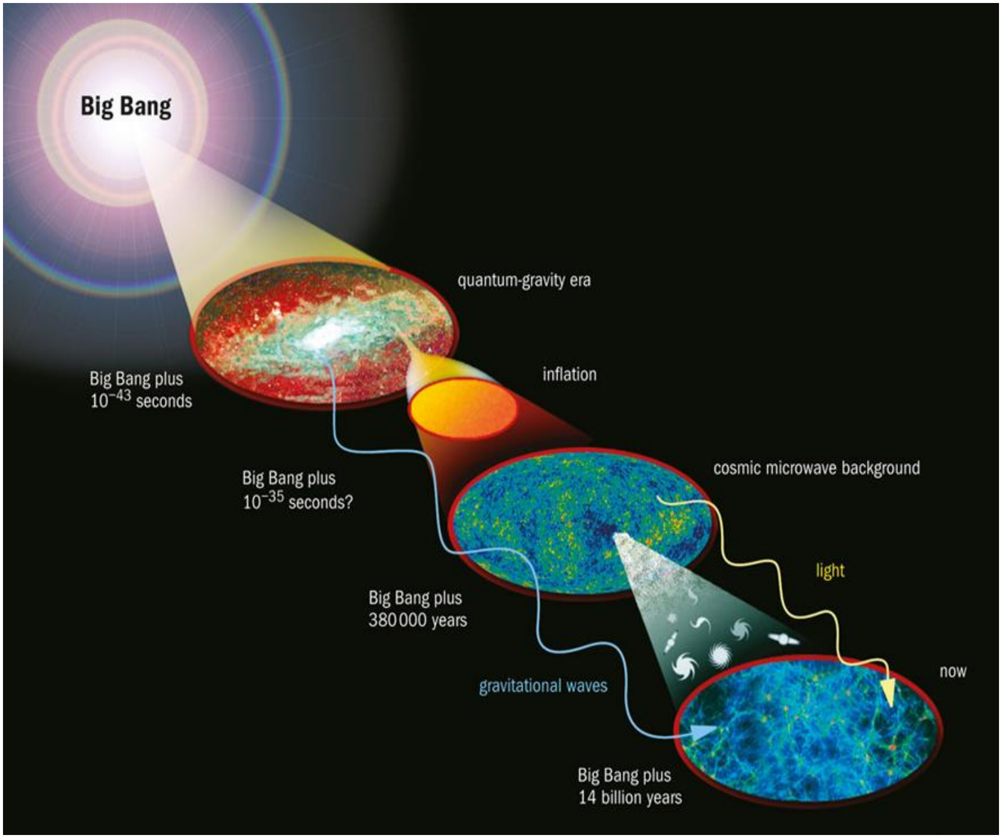
- Kazanas, Guth, Linde (1982): The Universe $\boxed{10^{-36}\,\mathrm{sec}}$ after the Big Bang, through some mechanism went into an exponential expansion up to $\boxed{10^{-32}\,\mathrm{sec}}$ increasing in size ~ $\boxed{10^{30}}$ times: Inflation.

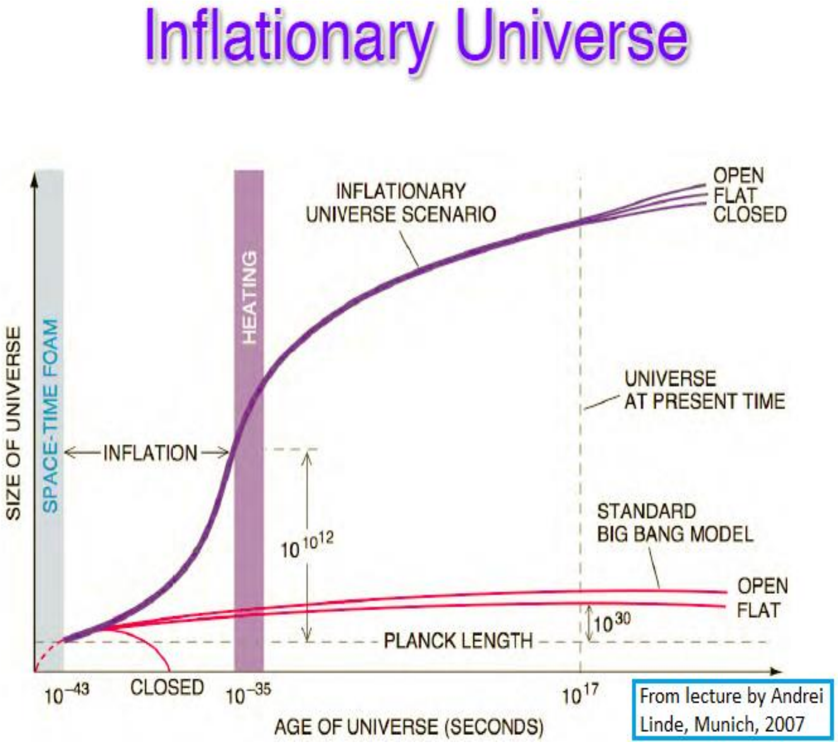
- I) The observable Universe is a tiny part of the total one, and originates from a small, causally connected region.

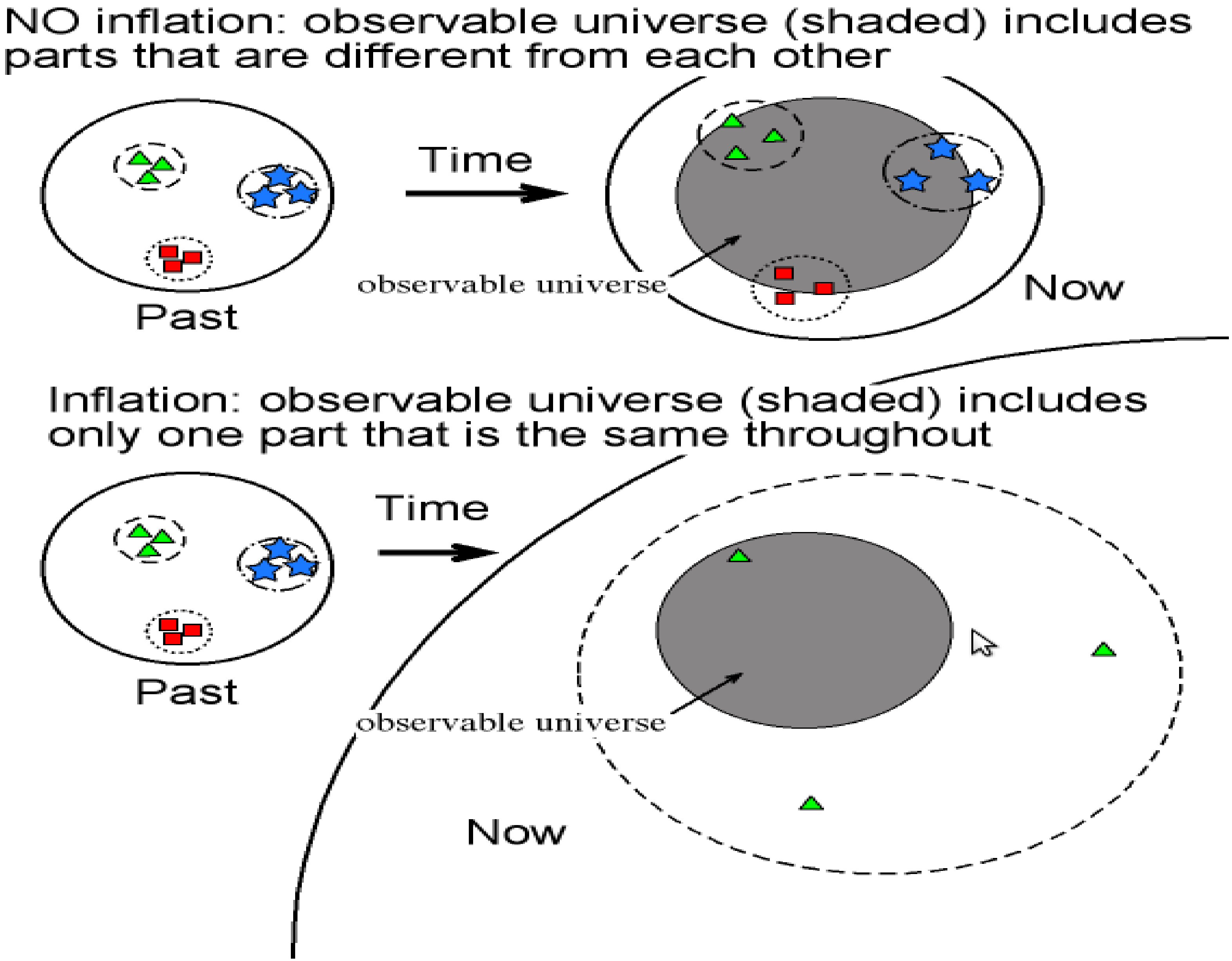
- II) Due to the huge expansion, the spatial curvature became almost zero.

- III) Due to the huge expansion the monopoles spread in all regions, and thus our own, observable universe, has at most one.

Inflation

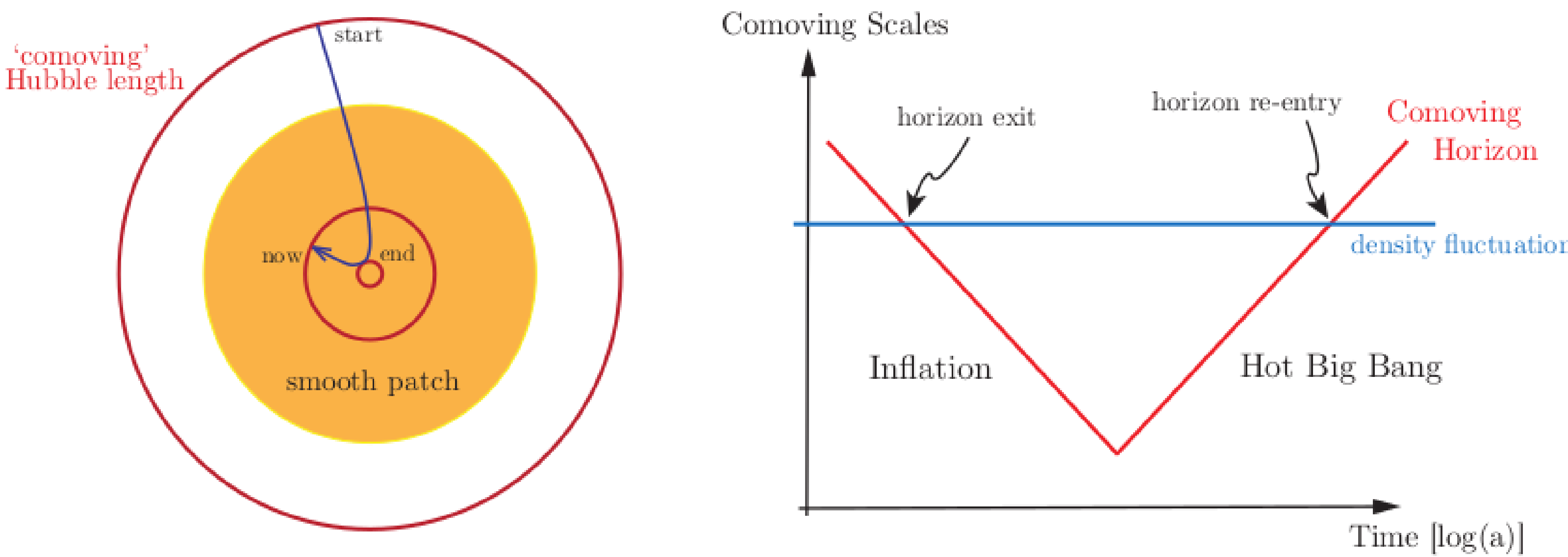

Big Bang
quantum-gravity era
Big Bang plus
10^{-43} seconds
inflation
Big Bang plus
10^{-35} seconds?
cosmic microwave background
light
Big Bang plus
380 000 years
gravitational waves
now
Big Bang plus
14 billion years


Inflationary Universe
INFLATIONARY
UNIVERSE SCENARIO
OPEN
FLAT
CLOSED
HEATING
SPACE-TIME FOAM
SIZE OF UNIVERSE
UNIVERSE
AT PRESENT TIME
INFLATION
$10^{10^{12}}$
STANDARD
BIG BANG MODEL
OPEN
FLAT
10^{30}
PLANCK LENGTH
10^{-43}
CLOSED
10^{-35}
10^{17}
AGE OF UNIVERSE (SECONDS)
From lecture by Andrei
Linde, Munich, 2007


NO inflation: observable universe (shaded) includes parts that are different from each other
Past
Time
observable universe
Now
Inflation: observable universe (shaded) includes only one part that is the same throughout
Past
Time
observable universe
Now

Horizon Problem Revisited

A decreasing comoving horizon means that large scales entering the present universe were inside the horizon before inflation Causal physics before inflation therefore established spatial homogeneity. With a period of inflation, the uniformity of the CMB is not a mystery.

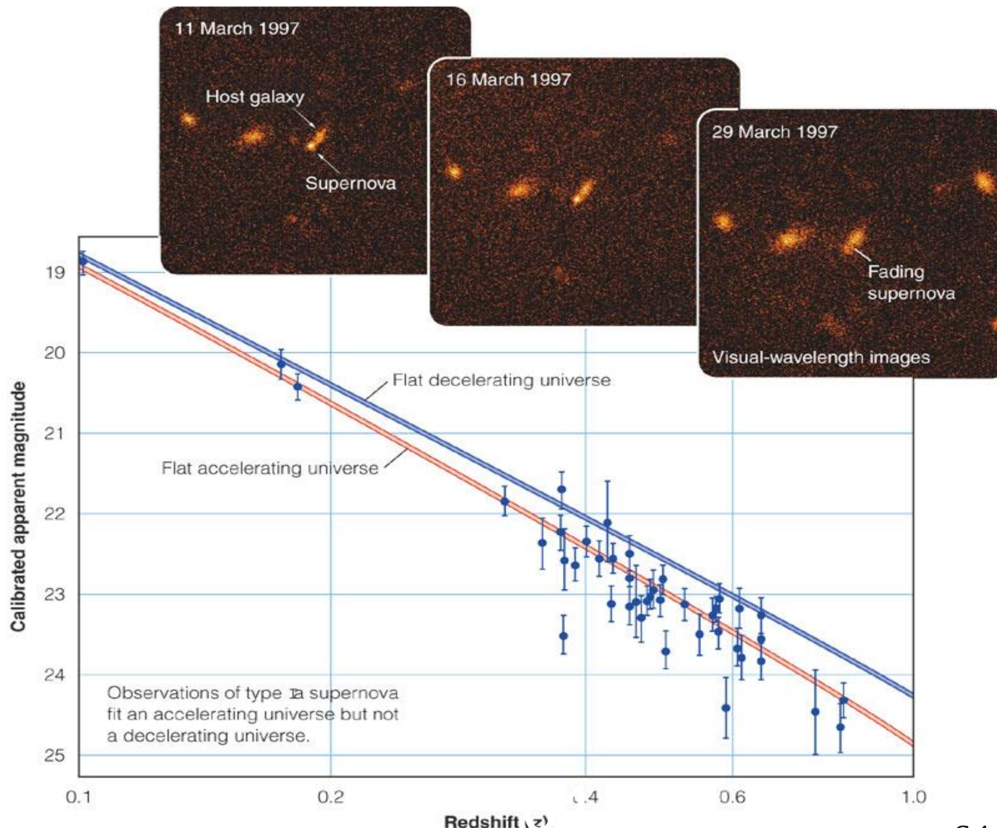


Left: Evolution of the comoving Hubble radius, $(aH)^{-1}$, in the inflationary universe. The comoving Hubble sphere shrinks during inflation and expands after inflation. Inflation is therefore a mechanism to 'zoom-in' on a smooth sub-horizon patch. *Right:* Solution of the horizon problem. All scales that are relevant to cosmological observations today were larger than the Hubble radius until $a \sim 10^{-5}$. However, at sufficiently early times, these scales were smaller than the Hubble radius and therefore causally connected. Similarly, the scales of cosmological interest came back within the Hubble radius at relatively recent times.

Dark Energy

- The Supernovae type Ia (explosions of binaries with one being white dwarf) are "standard candles", since their absolute magnitude M can be determined.

- In 1998 oı Perlmutter, Schmidt, Riess observed that 50 SnIa had smaller apparent magnitude than expected hence light traveled more, and thus the Universe today expands faster than before!



Dark Energy

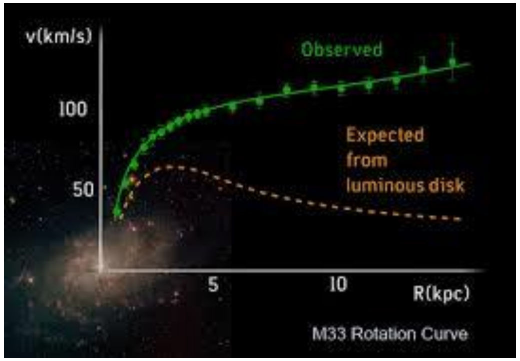
- The accelerated expansion is verified by independent observations, Cosmic Microwave Background (CMB), Baryon Acoustic Oscillations (BAO), Large Scale Structure (LSS), etc

- Around 70% of the total energy density of the Universe is this unknown dark energy (it does not interact electromagnetically).

- Possible explanation: The cosmological constant Λ (Einstein's "greatest blunder"). A term that produces the extra "repulsion".

Dark Matter

- Galaxy rotation curves:



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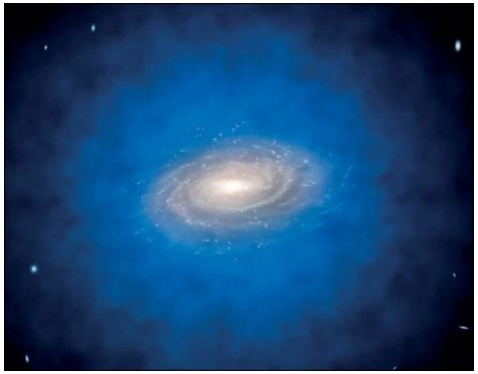
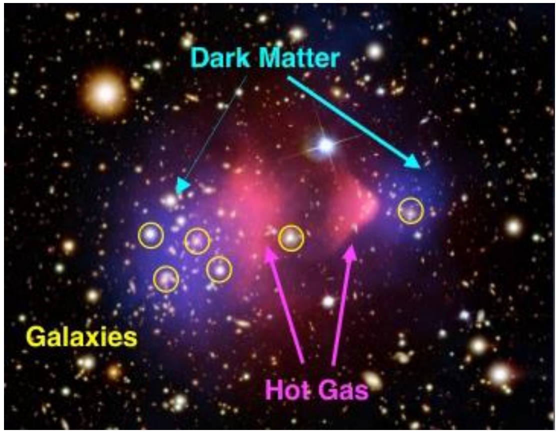


- Bullet cluster (collision of two galaxy clusters)



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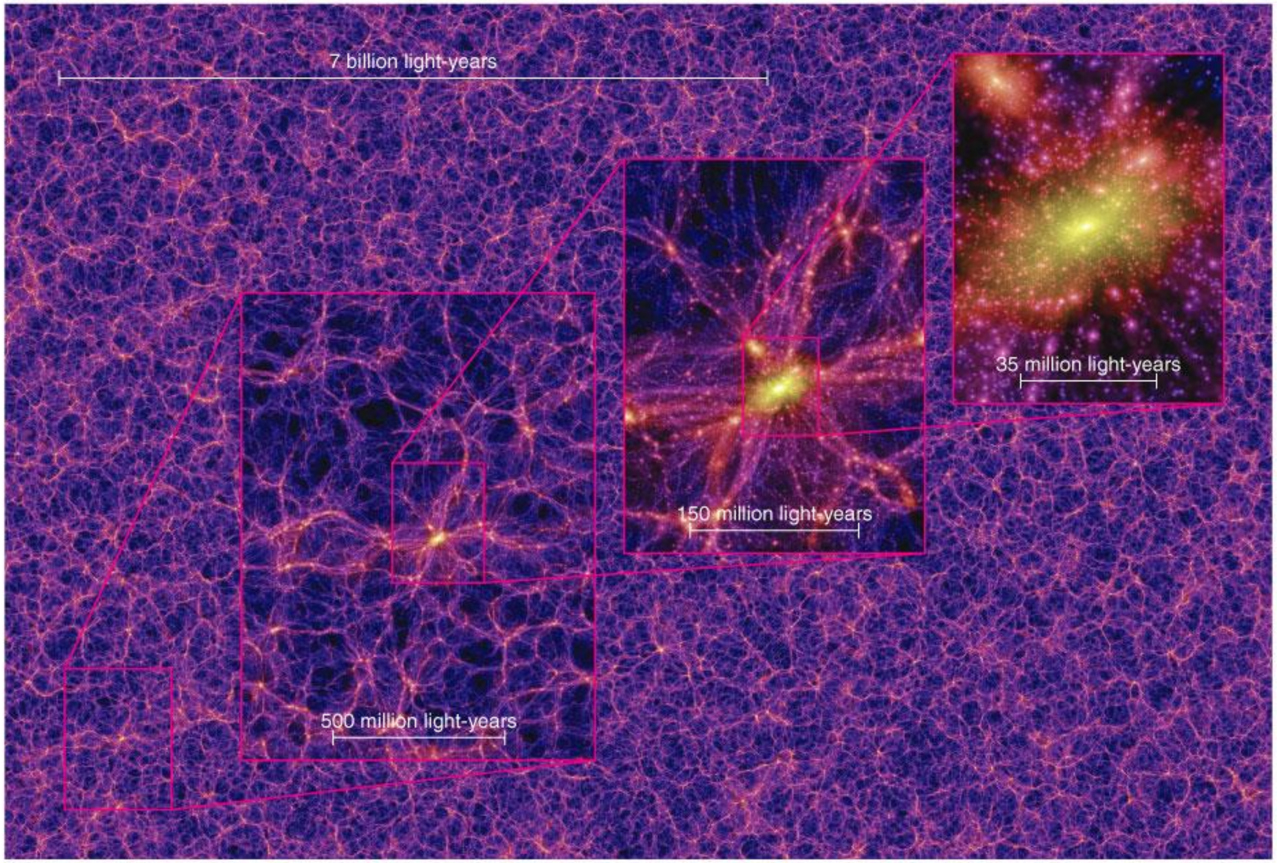
- 80% of matter is an "unknown" dark matter (it does not interact electromagnetically)!

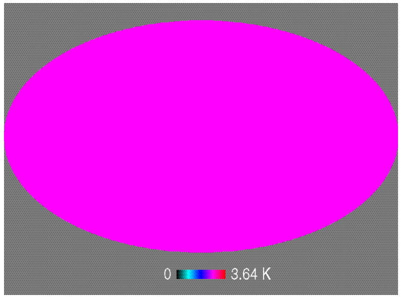
Dark Matter



7 billion light-years
500 million light-years
150 million light-years
35 million light-years

Cosmic Microwave Background radiation

- Since 1989, COBE, WMAP και Planck satellites show that CMB has small fluctuations:



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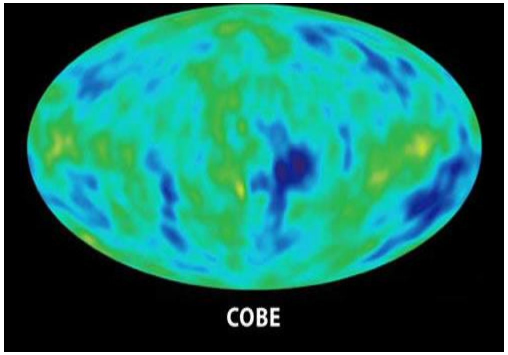

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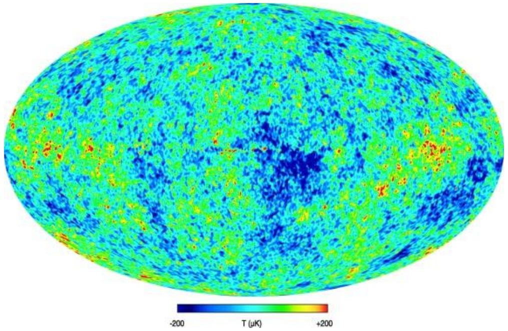

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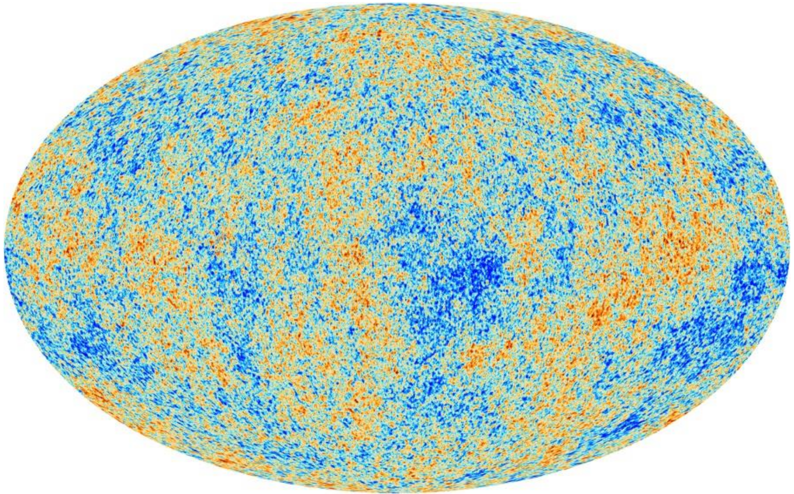
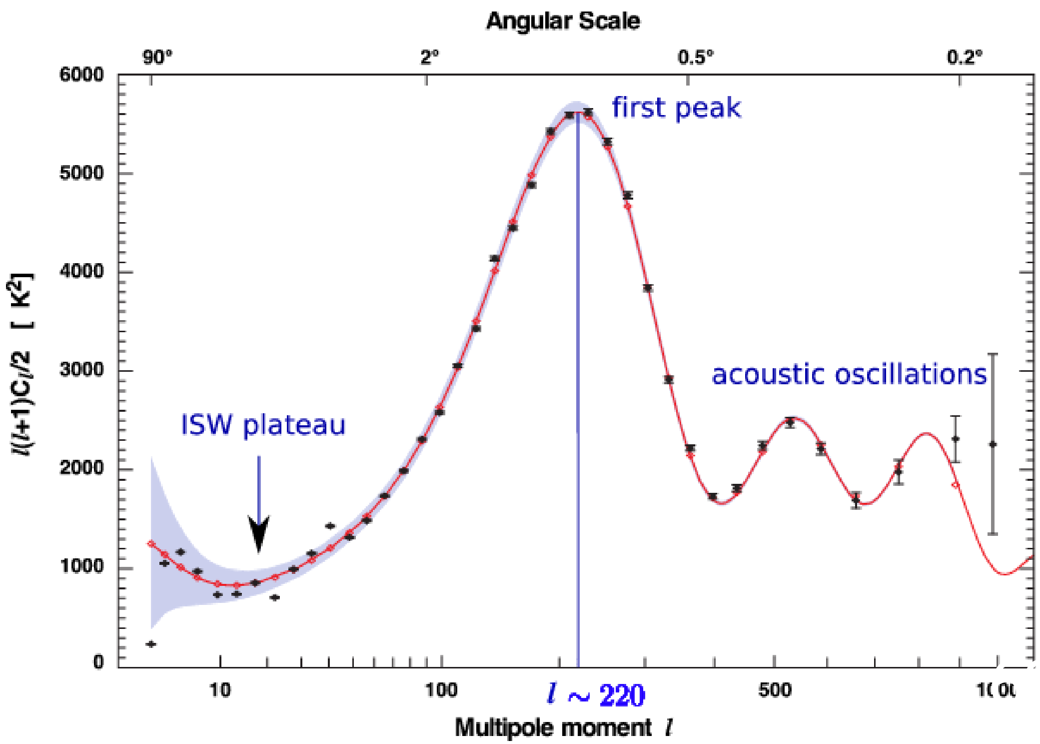


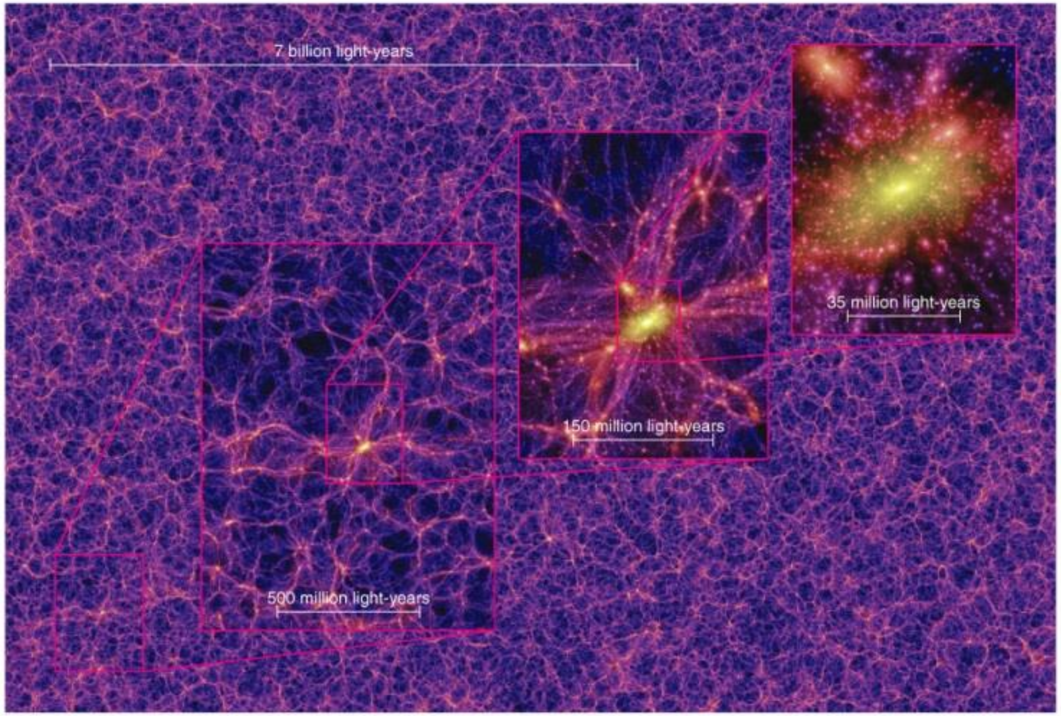
Cosmic Microwave Background radiation

- From the fluctuation spectrum we extract information: The first peak provides the spatial curvature (it results to flat universe), the second peak the baryon energy density parameter, the third peak the dark matter energy density parameter, etc.



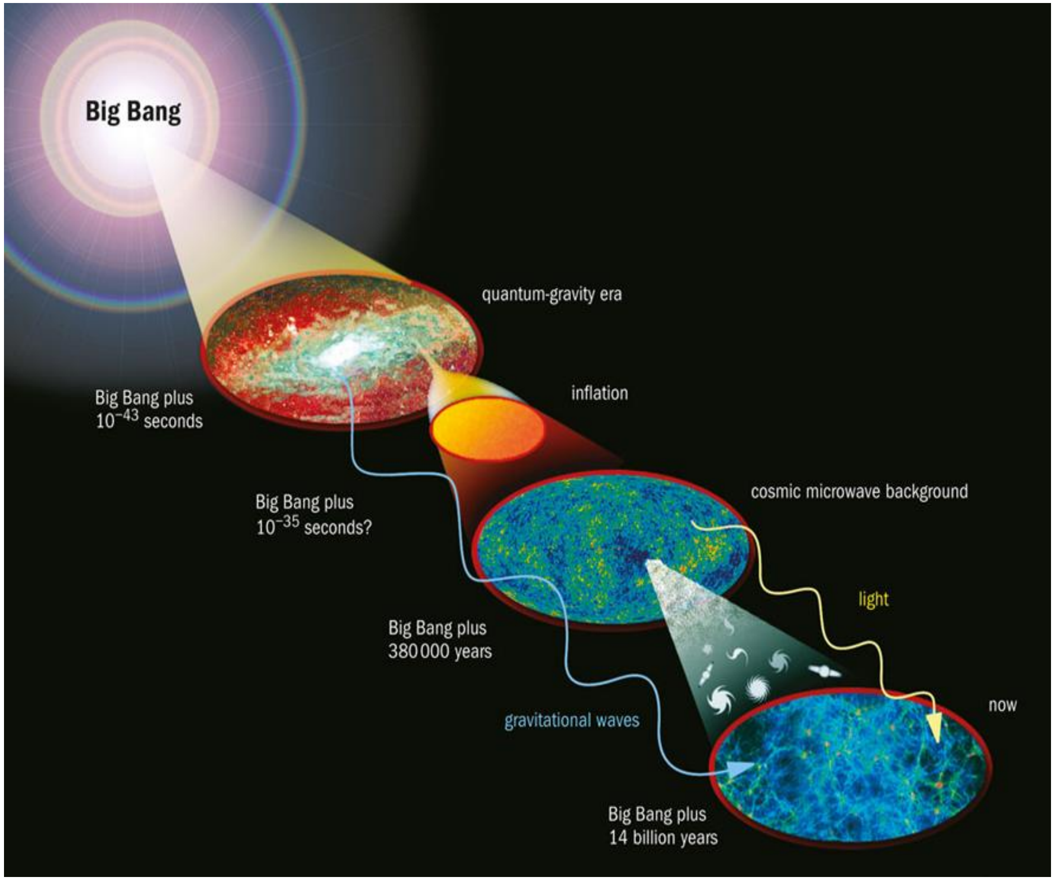
Inflation can also explain CMB and seeds of LSS

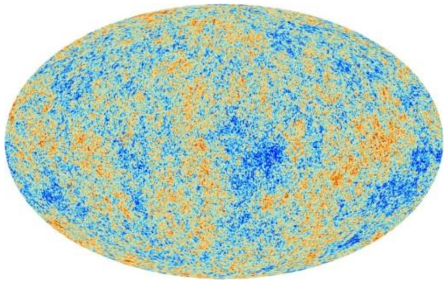
- Additional success: Inflation provides the necessary primordial fluctuations, which letter gave the Large Scale Structure of matter:

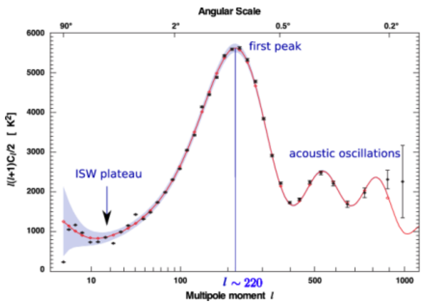


Amerigo Vespuci - 1499

Cosmic Inflation

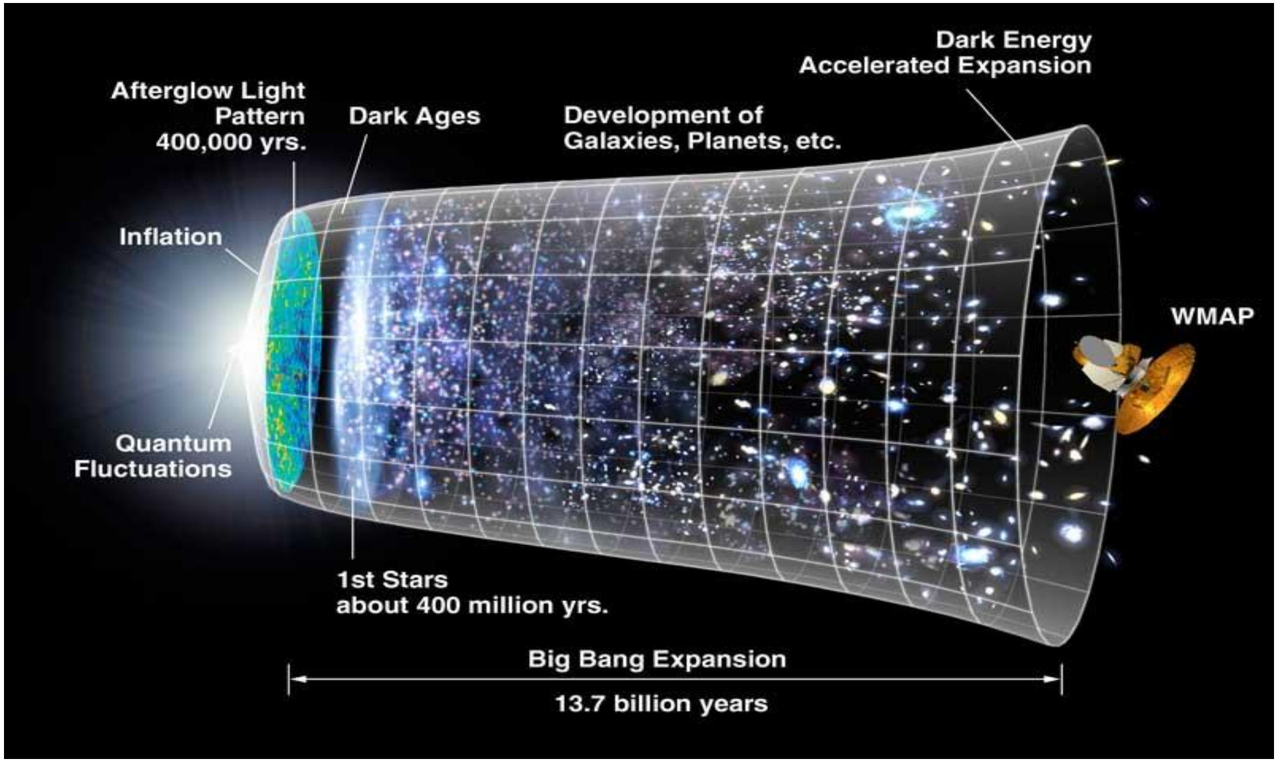


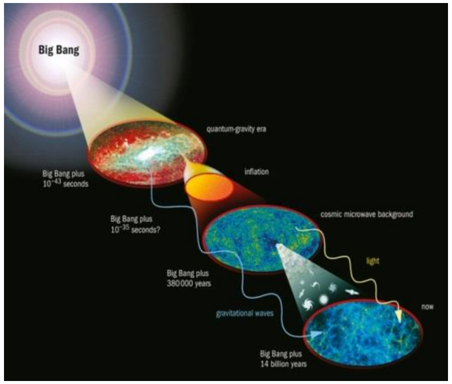


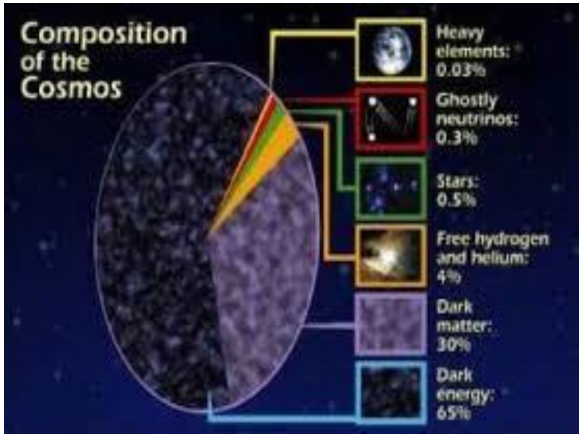


Summary of Observations

The Universe history:

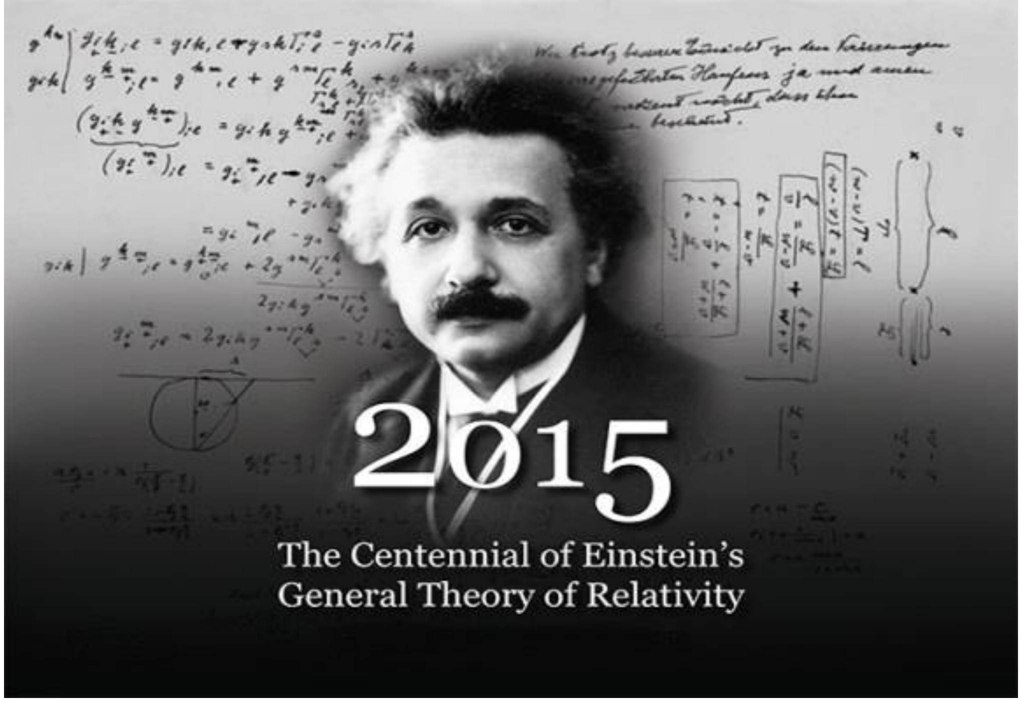






How to describe the Expanding Universe?

- General Relativity: The evolution of the 4-dimensional spacetime is determined by the distribution of matter



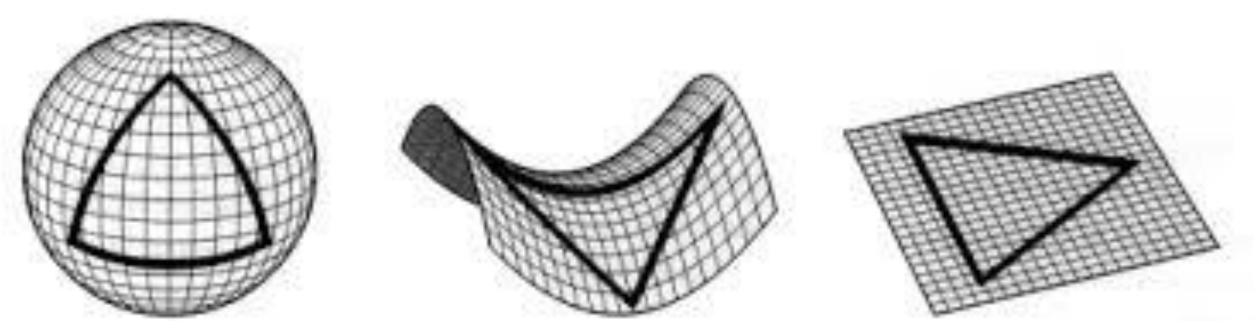
Describing Expanding Universe

- Homogeneous and Isotropic (Friedman-Robertson-Walker metric):

$$\boxed{ds^2 = -c^2 dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \right)}$$

$a(t)$: scale factor,

k=0,-1,+1 flat, closed open 3D spatial geometry



Positive Curvature   Negative Curvature   Flat Curvature

Describing Expanding Universe

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{16\pi G}\right]+S_m$$

- Field equations in FRW geometry (Friedmann Equations):

$$\boxed{H(t)^2+\frac{k}{a(t)^2}=\frac{8\pi G}{3}\rho(t)}$$

$$\boxed{\dot{H}(t)-\frac{k}{a(t)^2}=-4\pi G\left[\rho(t)+p(t)\right]}$$

$$H(t)\equiv\frac{\dot{a}(t)}{a(t)}$$

- Conservation Equation of matter perfect fluid

$$\boxed{\dot{\rho}(t)+3H(t)\left[\rho(t)+p(t)\right]=0}$$

Describing Expanding Universe

- Equation of State: $$\boxed{w \equiv \frac{p}{\rho}}$$

- Evolution of the universe for a fluid with constant w,
  in flat space (k=0):

$$\dot{\rho}(t) + 3H(t)\left[\rho(t) + p(t)\right] = 0 \qquad \boxed{\Rightarrow \quad \rho(t) = \rho_0 a^{-3(1+w)}}$$

$$H(t)^2 = \frac{8\pi G}{3}\rho(t) \qquad \boxed{\Rightarrow \quad a(t) = \left[\frac{3(1+w)}{2}\sqrt{\frac{8\pi G\rho_0}{3}}\right]^{\frac{2}{3(1+w)}} t^{\frac{2}{3(1+w)}}}$$

- Matter Universe ($w_m = 0$): $\boxed{a(t) \propto t^{2/3}}$
- Radiation Universe ($w_r = 1/3$): $\boxed{a(t) \propto t^{1/2}}$

Standard Model of Cosmology

ΛCDM Paradigm + Inflation

$$\boxed{H(t)^2 + \frac{k}{a(t)^2} = \frac{8\pi G}{3}\left[\rho_{dm}(t) + \rho_b(t) + \rho_r(t)\right] + \frac{\Lambda}{3}}$$

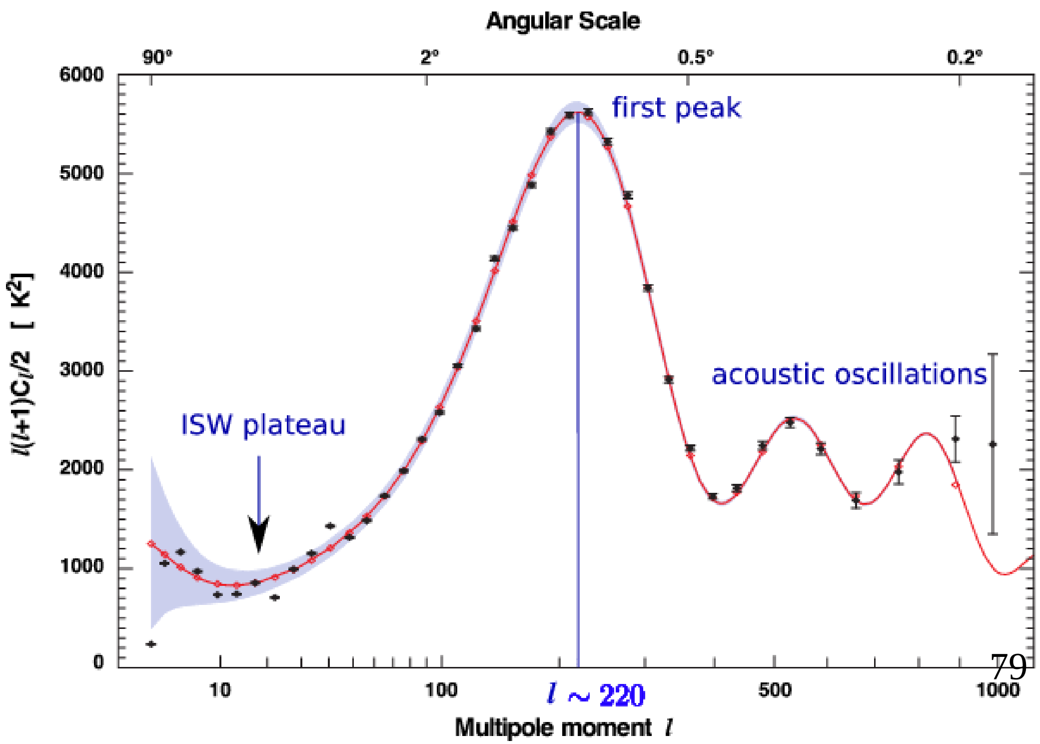
$$\boxed{w_\Lambda \equiv \frac{p_\Lambda}{\rho_\Lambda} = -1}$$

$$\boxed{\dot{H}(t) - \frac{k}{a(t)^2} = -4\pi G\left[\rho_{dm}(t) + p_{dm}(t) + \rho_b(t) + p_b(t) + \rho_r(t) + p_r(t)\right]}$$

- Describes the thermal history of the Universe at the background level
- Epochs of inflation, radiation, matter, late-time acceleration

Cosmic Microwave Background radiation

- From the fluctuation spectrum we extract information: The first peak provides the spatial curvature (it results to flat universe), the second peak the baryon energy density parameter, the third peak the dark matter energy density parameter, etc.



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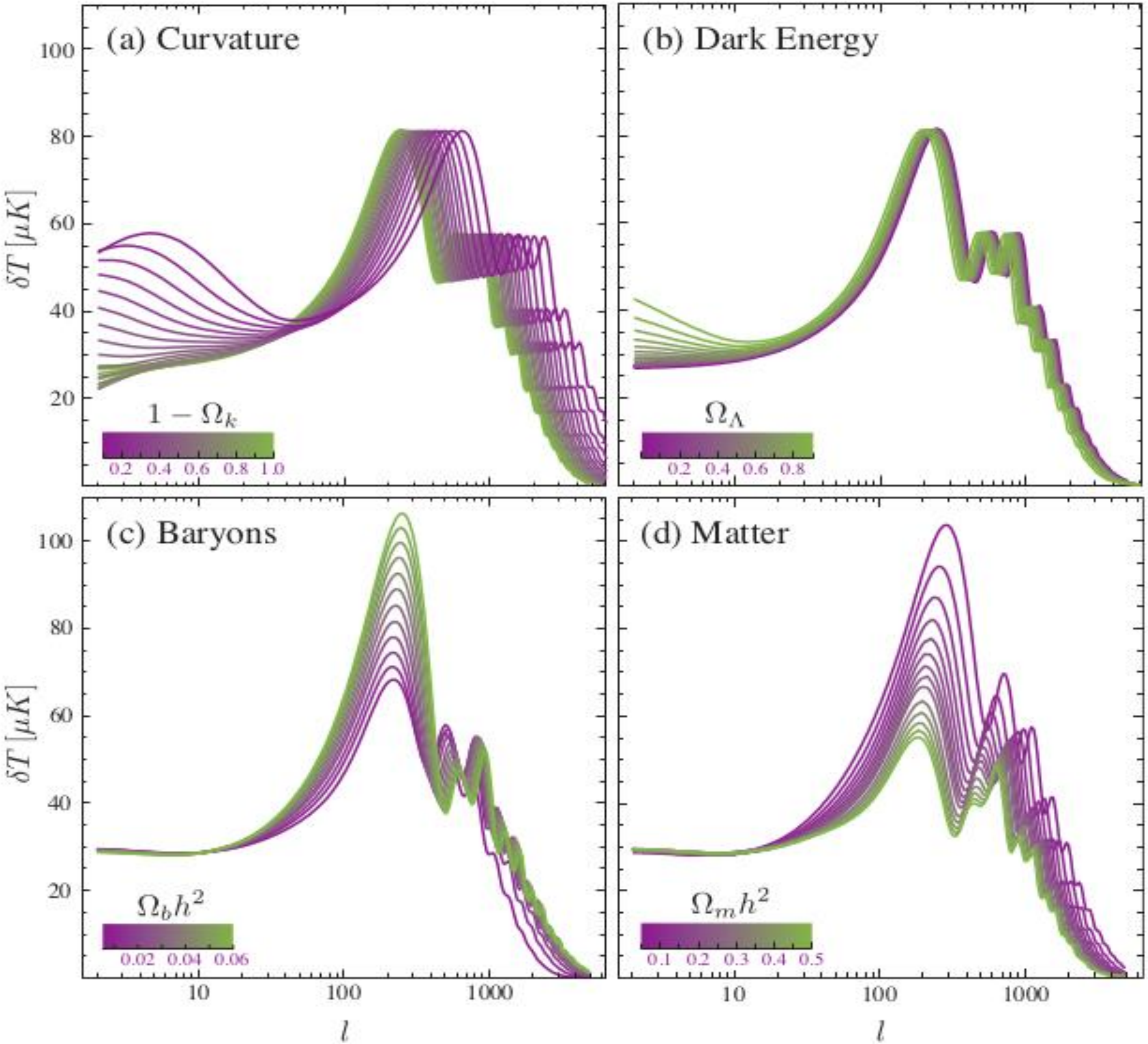
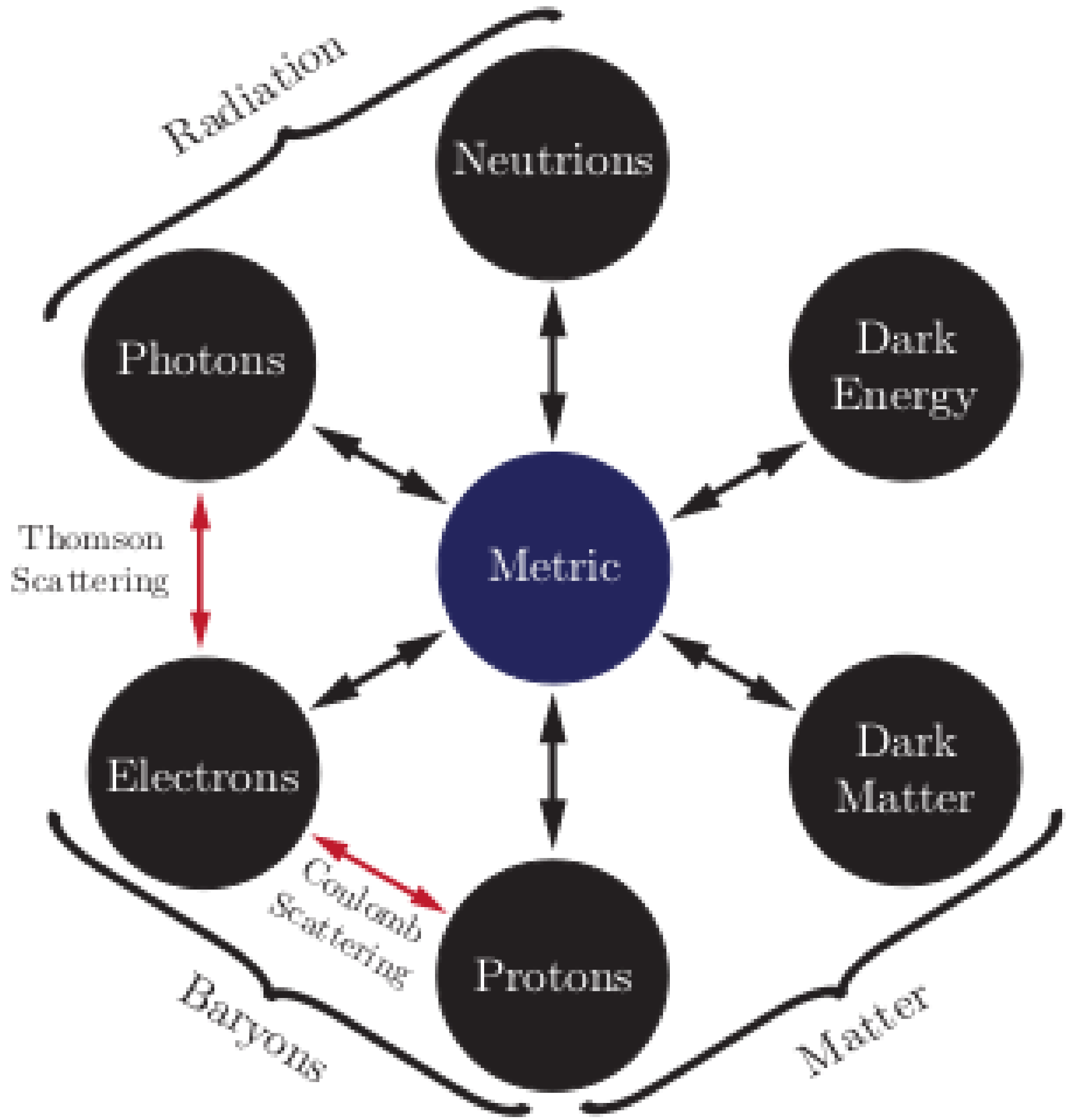


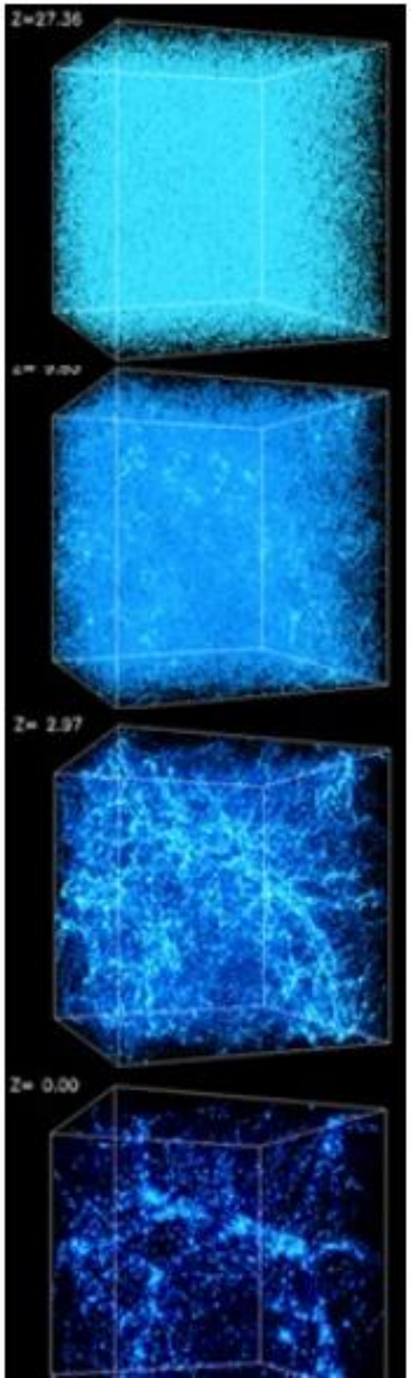
Figure 26. The CMB power spectrum as a function of cosmological parameters

LARGE SCALE STRUCTURE



Radiation
Neutrinos
Photons
Dark
Energy
Metric
Thomson
Scattering
Electrons
Dark
Matter
Coulomb
Scattering
Protons
Baryons
Matter

Evolution of the LSS – a brief history



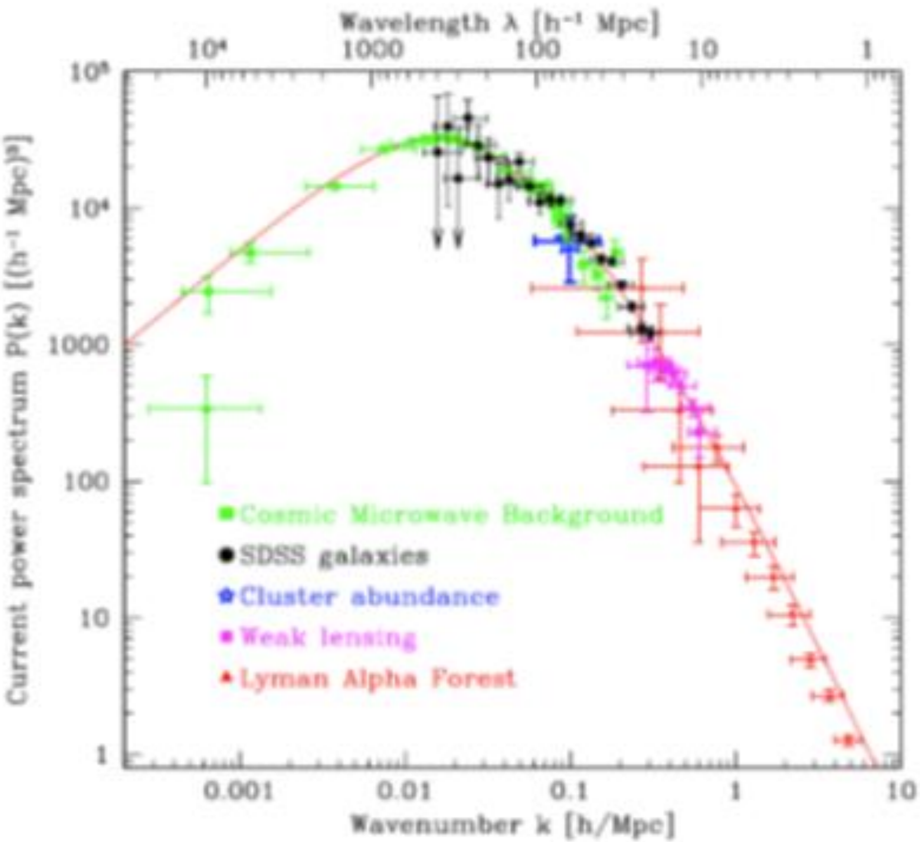
Somewhat after recombination --
density perturbations are small on nearly all spatial scales.

Dark Ages, prior to z=10 --
density perturbations in dark matter and baryons grow;
on smaller scales perturbations have gone non-linear, δ>>1;
small (low mass) dark matter halos form; massive stars
form in their potential wells and reionize the Universe.

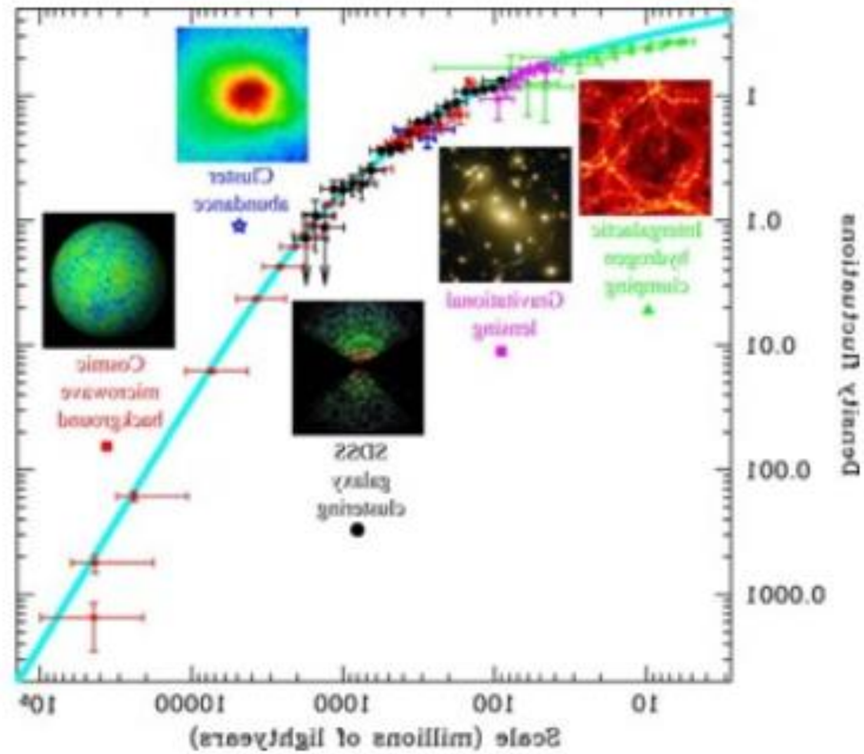
z=2 --
Most galaxies have formed; they are bright with stars;
this is also the epoch of highest quasar activity;
galaxy clusters are forming. In LCDM growth of structure
on large (linear) scales has nearly stopped, but smaller
non-linear scales continue to evolve.

z=0 --
Small galaxies continue to get merged to form larger ones;
in an open and lambda universes large scale (>10-100Mpc)
potential wells/hill are decaying, giving rise to late ISW.

Matter Density Fluctuation Power Spectrum



A different convention:
plot $P(k)k^3$



- From the conservation of the stress-tensor, we derived the relativistic generalisations of the continuity equation and the Euler equation

$$\delta' + 3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta = -\left(1 + \frac{\bar{P}}{\bar{\rho}}\right)\left(\nabla\cdot\boldsymbol{v} - 3\Phi'\right) , \tag{4.4.173}$$

$$\boldsymbol{v}' + 3\mathcal{H}\left(\frac{1}{3} - \frac{\bar{P}'}{\bar{\rho}'}\right)\boldsymbol{v} = -\frac{\nabla\delta P}{\bar{\rho} + \bar{P}} - \nabla\Phi . \tag{4.4.174}$$

These equations apply for the total matter and velocity, and also separately for any non-interacting components so that the individual stress-energy tensors are separately conserved.

- A very important quantity is the comoving curvature perturbation

$$\mathcal{R} = -\Phi - \frac{\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{4\pi G a^2(\bar{\rho} + \bar{P})} . \tag{4.4.175}$$

We have shown that $\mathcal{R}$ doesn't evolve on super-Hubble scales, $k \ll \mathcal{H}$, unless non-adiabatic pressure is significant.

SVT decomposition

A very important result of General Relativity is the *Scalar, Vector and Tensor decomposition theorem*: each type of metric perturbation evolves *independently* at *linear order*.

Putting each type of perturbation together we get:

$$
\begin{aligned}
S &: ds^2 = -(1+2\Phi)dt^2 + 2\alpha(t)B_{,i}dx^i dt + \alpha^2(t)[(1-2\Psi)\delta_{ij} + 2E_{,ij}]dx^i dx^j \\
V &: ds^2 = -dt^2 + 2\alpha(t)B_i dx^i dt + \alpha^2(t)[\delta_{ij} + 2V_{(i,j)}]dx^i dx^j \\
T &: ds^2 = -dt^2 + \alpha^2(t)[\delta_{ij} + h_{ij}^{TT}]dx^i dx^j
\end{aligned} \tag{15}
$$

Note: We can perform a similar decomposition for $T_{\mu\nu}$, which is also a symmetric and rank 2 tensor.

Coordinate systems- Gauges

- General Relativity has diffeomorphism invariance $\rightarrow$ free choice of coordinate system or *gauge*.
- In certain systems some of the functions we introduced *gauge away*.
- We will first consider the *scalar* perturbations:

$$\begin{aligned} t \rightarrow \hat{t} &= t + \zeta(t, \vec{x}) \\ x^i \rightarrow \hat{x}^i &= x^i + \xi^{,i}(t, \vec{x}) \end{aligned} \tag{16}$$

- From (15):

$$\begin{aligned} g_{00} &= -(1 + 2\Phi) \\ g_{0i} &= -\alpha(t) B_{,i} \\ g_{ij} &= \alpha^2(t) \left[\delta_{ij}(1 - 2\Psi) + 2E_{ij}\right] \end{aligned}$$

Scalar Perturbations

- The metric is rank 2 tensor, thus it transforms as follows:

$$g_{\mu\nu}(x) = \frac{\partial \hat{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \hat{x}^{\beta}}{\partial x^{\nu}} g_{\hat{\alpha}\beta}(\hat{x}) \tag{17}$$

- The time- time component of (17) is:

$$-(1+2\Phi) = -(1+2\hat{\Phi})\left(\frac{\partial \hat{t}}{\partial t}\right)^2 \rightarrow -(1+2\Phi) = -(1+2\hat{\Phi})(1+\dot{\zeta})^2$$

$$\approx -1 - 2\hat{\Phi} - 2\dot{\zeta} \rightarrow \quad \hat{\Phi} = \Phi - \dot{\zeta}$$

Scalar Perturbations

- Similarly we can work out the transformation of the other functions and find:

$$\begin{aligned}\hat{\Phi} &= \Phi - \dot{\zeta} \\ \hat{B} &= B - \zeta/\alpha + \alpha\dot{\xi} \\ \hat{E} &= E - \xi \\ \hat{\Psi} &= \Psi + H\zeta \end{aligned} \tag{18}$$

- Are all these perturbation functions necessary $\Leftrightarrow$ are these perturbations *fictitious* or real ? $\rightarrow$ Need to construct gauge invariants:

$$Bardeen \begin{cases} \Phi_B & \equiv \Phi - \frac{d}{dt}\left[\alpha^2(\dot{E} - B/\alpha)\right] \\ \Psi_B & \equiv \Psi + \alpha^2 H(\dot{E} - B/\alpha) \end{cases} \tag{19}$$

Matter Perturbations

- From (23) we deduce that there are exactly *two* (4 - 2 = 2) such independent gauge invariant combinations.
- To actually describe the *structure* in the universe $\rightarrow$ need *matter* perturbations as well.
- We will consider perturbations around the homogeneous energy momentum tensor:

$$\bar{T}^{\mu}_{\nu} = (\bar{\rho} + \bar{p})\bar{u}^{\mu}\bar{u}_{\nu} + \bar{p}\delta^{\mu}_{\nu} \tag{20}$$

and write them as follows:

Matter Perturbations

$$
\begin{aligned}
T^0_0 &= -(\bar{\rho} + \delta\rho) \\
T^0_i &= (\bar{\rho} + \bar{p})\alpha\upsilon_i \\
T^i_0 &= -(\bar{\rho} + \bar{p})(\upsilon^i - B^i)/\alpha \\
T^i_j &= \delta^i_j(\bar{p} + \delta p) + \Sigma^i_j
\end{aligned} \tag{21}
$$

where $\upsilon^i \equiv dx^i/d\tau$ and Σ^i_j is the anisotropic stress.
For each different universe constituent we have:

$$
\begin{aligned}
\delta\rho &= \sum_I \delta\rho_I & \delta p &= \sum_I \delta p_I \\
(\bar{\rho} + \bar{p})\upsilon^i &= \sum_I (\bar{\rho}_I + \bar{p}_I)\upsilon^i_I & \Sigma^{ij} &= \sum_I \Sigma^{ij}_I
\end{aligned} \tag{22}
$$

Matter Perturbations

- Velocities do not simply add $\rightarrow$ define $\delta q^i \equiv (\bar{\rho} + \bar{p})\alpha v^i$ which is the 3- momentum density and $\delta q^i = \sum_I \delta q^i_I$
- A scalar function $\Phi(x)$ under (16) becomes:

$$\hat{\Phi}(\hat{x}) = \Phi(\hat{t} - \zeta, \hat{x}^i - \xi^{,i}) = \bar{\Phi}(\hat{t} - \zeta) + \delta\Phi(\hat{t} - \zeta, \hat{x}^i - \xi^{,i})$$
$$\approx \bar{\Phi}(\hat{t}) - \zeta\frac{d\bar{\Phi}}{d\hat{t}} + \delta\Phi(\hat{t}, \hat{x}^i) \Rightarrow \delta\hat{\Phi} = \delta\Phi - \zeta\frac{d\bar{\Phi}}{dt} \qquad (23)$$

- From (23) and the tensor transformation law we get:

$$\delta\hat{\rho} = \delta\rho - \bar{\rho}\zeta$$
$$\delta\hat{p} = \delta p - \bar{p}\zeta$$
$$\delta\hat{q} = \delta q + (\bar{\rho} + \bar{p})\zeta \qquad (24)$$

where q is the scalar part of the momentum density.

Matter Perturbations

- Given (24) we can construct *gauge invariant* quantities like:
  i) The *comoving density* perturbation:

$$\delta\rho_m \equiv \delta\rho - 3H\delta q \tag{25}$$

  ii) The *curvature perturbation on uniform density hypersurfaces*:

$$-\zeta \equiv \Psi + H\delta\rho/\dot{\bar{\rho}} \tag{26}$$

  iii) The *comoving curvature perturbation* :

$$\mathcal{R} = \Psi - \frac{H}{\bar{\rho} + \bar{p}}\delta q \tag{27}$$

- The Fourier transformation is: $f(\vec{x}) = \int \frac{d^3x}{(2\pi)^3}\, \tilde{f}_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}}$

Einstein Equations

- Can relate the metric and stress-energy perturbations via the *perturbed* Einstein equations:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \tag{28}$$

which to linear order and for *scalar* perturbations in Fourier space yields:

- *Energy and momentum constraint equations:*

$$\begin{aligned} 3H(\dot{\Psi} + H\Phi) + \frac{k^2}{\alpha^2}\left[\Psi + H(\alpha^2\dot{E} - \alpha B)\right] &= -4\pi G \delta\rho \\ \dot{\Psi} + H\Phi &= -4\pi G \delta q \end{aligned} \tag{29}$$

- From (19), (25), (29) $\rightarrow$ gauge-invariant Poisson Equations:

$$\frac{k^2}{\alpha^2}\Psi_B = -4\pi G \delta\rho_m \tag{30}$$

Einstein Equations

- From (29) we can also get the *evolution equations*:

$$\ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi} + (3H^2 + 2\dot{H})\Phi = -4\pi G\big(\,\delta p - 2k^2\delta\Sigma/3\big)$$

$$(\partial_t + 3H)(\dot{E} - B/\alpha) + \frac{\Psi - \Phi}{\alpha^2} = 8\pi G\,\delta\Sigma \tag{31}$$

- From (19) and (49) we can write:

$$\Psi_B - \Phi_B = 8\pi G\alpha^2\,\delta\Sigma \tag{32}$$

Note: From (32) if $\delta\Sigma \approx 0 \;\Rightarrow\; \Psi_B \approx \Phi_B$

Conservation Equations

- $\nabla_\mu T^{\mu\nu} = 0 \rightarrow$ *continuity equation* and *Euler equation* :

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) = \frac{k^2}{\alpha^2}\delta q + (\bar{\rho} + \bar{p})\left[3\dot{\Psi} + k^2(\dot{E} - B/\alpha)\right] \quad (33)$$

$$\dot{\delta q} + 3H\delta q = -\delta p - (\bar{\rho} + \bar{p})\Phi + 2k^2\delta\Sigma/3 \quad (34)$$

Note: It would be useful to note the 0th version of (33):

$$\dot{\bar{\rho}} = -3H(\bar{\rho} + \bar{p}) \quad (35)$$

We are almost ready to pick a gauge and do explicit calculations

$\rightarrow$ need *initial conditions* for our perturbations.

Initial Conditions

- It is standard to assume that these perturbations are generated from *inflation*, which predicts that they are *isentropic or adiabatic* - also preferred by the data.

- Adiabatic perturbations are induced by a common, local shift in time of all background quantities so:

$$\delta\eta = \frac{\delta\rho_a}{\bar{\rho}'_a} = \frac{\delta\rho_b}{\bar{\rho}'_b} \quad \text{for each species a and b} \tag{36}$$

where conformal time is $dn \equiv dt/\alpha$ and from (35) :

$$\frac{\delta_a}{1+w_a} = \frac{\delta_b}{1+w_b} \tag{37}$$

where we defined $\delta \equiv \delta\rho/\rho$ the *fractional overdensity* and $w_a \equiv P_a/\rho_a$ the equation of state parameter.

Newtonian Gauge

We can now start describing the evolution of structure. We shall pick the Newtonian gauge for it simplifies the analytic calculations greatly.

Newtonian gauge: *Definition:* $B = E = 0$

The equations we have presented become:

$$ds^2 = -(1+2\Phi)dt^2 + \alpha^2(t)[(1-2\Psi)\delta_{ij}]dx^i dx^j \tag{38}$$

i) The Einstein equations:

$$\begin{aligned} 3H(\dot{\Psi} + H\Phi) + \frac{k^2}{\alpha^2}\Psi &= -4\pi G\,\delta\rho \\ \dot{\Psi} + H\Phi &= -4\pi G\,\delta q \\ \ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi} + (3H^2 + 2\dot{H})\Phi &= -4\pi G\big(\,\delta p - 2k^2\delta\Sigma/3\big) \\ \frac{\Psi - \Phi}{\alpha^2} &= 8\pi G\,\delta\Sigma \end{aligned} \tag{39}$$

Newtonian Gauge

The gauge invariant Poisson equation (30) becomes:

$$\frac{k^2}{\alpha^2}\Psi = -4\pi G\delta\rho \tag{40}$$

ii) The conservation equations:

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) = \frac{k^2}{\alpha^2}\delta q + 3(\bar{\rho} + \bar{p})\dot{\Psi} \tag{41}$$

$$\dot{\delta q} + 3H\delta q = -\delta p - (\bar{\rho} + \bar{p})\Phi + 2k^2\delta\Sigma/3 \tag{42}$$

It is useful to make (41) and (42) more explicit:

$$\begin{aligned} \dot{\delta} + 3H\Big(\frac{\delta\bar{P}}{\bar{\rho}} - \frac{\delta\bar{P}}{\bar{\rho}}\Big)\delta &= -\Big(1 + \frac{\bar{P}}{\bar{\rho}}\Big)\Big(\frac{k_i u^i}{\alpha} - 3\dot{\Phi}\Big) \\ \dot{u}^i + 3H\Big(\frac{1}{3} - \frac{\dot{\bar{P}}}{\dot{\bar{\rho}}}\Big)u^i &= -\frac{1}{\bar{\rho} + \bar{P}}\Big(\frac{k^i\delta P}{\alpha} + k_j\Sigma^{ij}\Big) - k^i\Psi \end{aligned} \tag{43}$$

Growth Equations

We will now consider inhomogeneities in a fluid with $w = \bar{P}/\bar{\rho}, \Sigma_{ij} = 0$, $c_s^2 \equiv \delta P/\delta\rho$ and *adiabatic* perturbations $\rightarrow c_s^2 \approx w$

Under these assumptions equations (43) become:

$$\dot{\delta} = -(1+w)\Big(\frac{k_i}{\alpha}u^i - 3\dot{\Psi}\Big) \tag{44}$$

$$\dot{u}^i = -H(1-3w)u^i - \frac{c_s^2}{1+w}\frac{k^i}{\alpha}\delta - \frac{k^i}{\alpha}\Psi \tag{45}$$

On subhorizon scales, that is $k >> \alpha H$, $\dot{\Psi} \approx 0$ and by taking the divergence of (45) and the Poisson equation (30) we get:

$$\boxed{\ddot{\delta} + (2-3w)H\dot{\delta} + c_s^2\frac{k^2}{\alpha^2} = (1+w)4\pi G\bar{\rho}\delta} \tag{46}$$

Which is called the *Jeans equation*.

Growth Equations

If we now consider the *late time* evolution of perturbations ($\Rightarrow c_s \approx 0$) in the *matter domination era* ($w \approx 0$). Then (46) yields:

$$\boxed{\ddot{\delta} + 2H\dot{\delta} = 4\pi G\rho_M\,\delta} \tag{47}$$

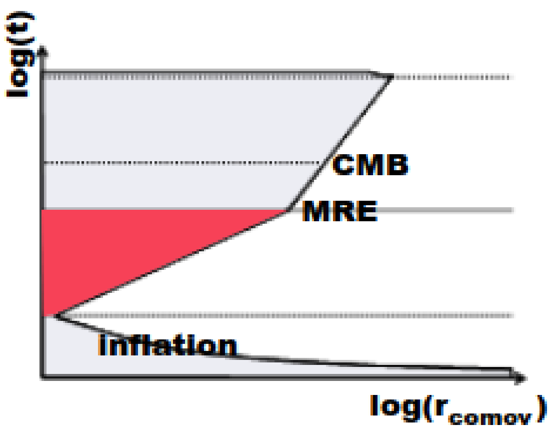
- By using this *linear* equation we can study the distribution of matter in the universe
- An analogous equation we can get in *modified gravity* theories, where the Newtonian G in (47) is replaced by G_{eff} .

Linear growth of density perturbations:

Sub-horizon, radiation dominated, pre recombination

Jeans linear perturbation analysis applies:

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left[\frac{c_s^2 k^2}{a^2} - 4\pi G\rho_0\right]\delta_k = 0$$



$$a \propto t^{1/2}$$

$$\dot{a} \propto \tfrac{1}{2} t^{-1/2}$$

$$\frac{\dot{a}}{a} = H = \frac{1}{2t}$$

$$Ht = \tfrac{1}{2}$$

radiation dominates, and because radiation does not cluster → all δ_k=0...

dark matter has no pressure of its owr it is not coupled to photons, so there no restoring pressure force.

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left[\underbrace{\cancel{\frac{c_s^2 k^2}{a^2}}}_{\text{zero}} - 4\pi G\rho_0\right]\delta_k = 0$$

$$\ddot{\delta}_k + \tfrac{1}{t}\dot{\delta}_k = 0$$

$$\delta_k = \underbrace{A\ln(t)}_{\text{growing mode}} + \underbrace{B}_{\text{“decaying” mode}}$$

...but the rate of chang of δ_k's can be non-zei

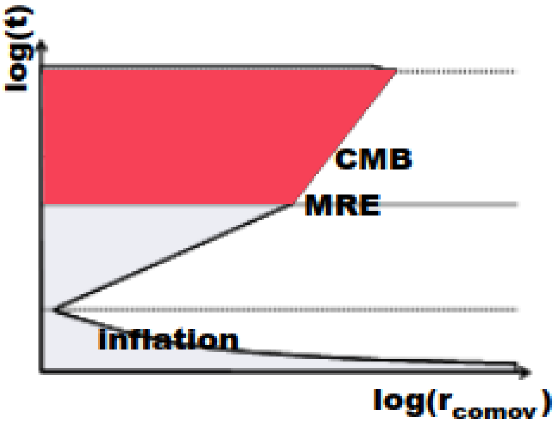
$$\boxed{\text{DM growing mode solution} \quad \delta_k \propto 2\ln(a)}$$

Linear growth of density perturbations:

Sub-horizon, matter dominated, pre & post recomb.

Jeans linear perturbation analysis applies:

$$\ddot{\delta}_k + 2H\dot{\delta}_k + [\tfrac{c_s{}^2 k^2}{a^2} - 4\pi G\rho_0]\,\delta_k = 0$$



$$a \propto t^{2/3}$$

$$\dot{a} \propto \tfrac{2}{3} t^{-1/3}$$

$$\frac{\dot{a}}{a} = H = \frac{2}{3t}$$

$$Ht = \tfrac{2}{3}$$

dark matter has no pressure of its own it is not coupled to photons, so there no restoring pressure force.

$$\ddot{\delta}_k + 2H\dot{\delta}_k + [\underbrace{\cancel{\tfrac{c_s{}^2 k^2}{a^2}}}_{\text{zero}} - 4\pi G\rho_0]\,\delta_k = 0$$

also, can assume that total density is the critical density at that epoch:

$$\rho = \rho_0 = \tfrac{3H^2}{8\pi G}$$

$$4\pi G\rho_0 = \tfrac{2}{3t^2}$$

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0$$

$$\delta_k = \underbrace{At^{2/3}}_{\text{growing mode}} + \underbrace{Bt^{-1}}_{\text{decaying mode}}$$

Two linearly indep. solutions: **growing** mode always comes to dominate; ignore **decaying** mode soln.

$$\boxed{\text{DM growing mode solution} \quad \delta_k \propto a}$$

Linear growth of density perturbations:
Sub-horizon, lambda dominated, pre & post recomb.

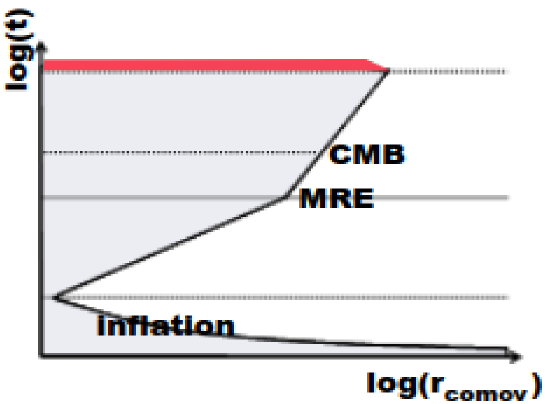
Jeans linear perturbation analysis applies:

$$\ddot{\delta}_k + 2H\dot{\delta}_k + [\frac{c_s^2 k^2}{a^2} - 4\pi G\rho_0]\,\delta_k = 0$$

$$H^2 = H_0^2[\Omega_\Lambda]$$
$$H = \text{const}$$
$$a \propto e^{Ht}$$

dark matter has no pressure of its own; it is not coupled to photons, so there is no restoring pressure force.

$$\ddot{\delta}_k + 2H\dot{\delta}_k + [\underbrace{\cancel{\frac{c_s^2 k^2}{a^2}}}_{\text{zero}} - 4\pi G\rho_0]\,\delta_k = 0$$



can assume the amplitude of perturbations is zero, because lambda, which dominates, does not cluster:

$$\delta_k = 0$$

$$\ddot{\delta}_k + 2H\dot{\delta}_k = 0$$

$$\delta_k = \underbrace{A}_{\text{"growing" mode}} + \underbrace{Be^{-2Ht}}_{\text{decaying mode}}$$

Two linearly indep. solutions: **growing** mode always comes to dominate; ignore **decaying** mode soln.

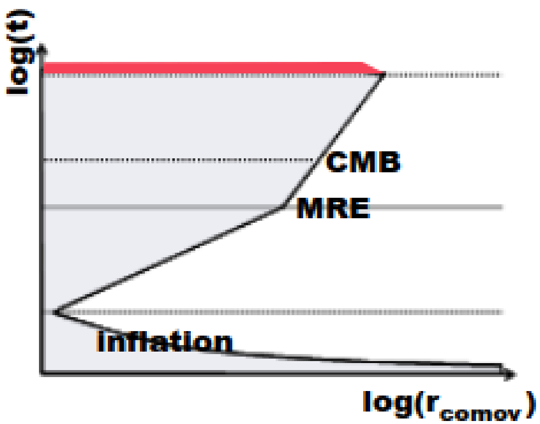
$$\boxed{\text{DM "growing" mode soln} \quad \delta_k \propto \text{const}}$$

Linear growth of density perturbations:

Sub-horizon, curvature dominated, pre & post recomb.

Jeans linear perturbation analysis applies:

$$\ddot{\delta}_k + 2H\dot{\delta}_k + [\frac{c_s^2 k^2}{a^2} - 4\pi G\rho_0]\,\delta_k = 0$$



$$H^2 = H_0^2[(1-\Sigma\Omega_w)a^{-2}]$$
$$H \propto a^{-1}$$
$$\dot{a} = \text{const}$$
$$a \propto t$$
$$Ht = 1$$

can assume the amplitude of perturbations is zero, because curvature, which dominates, does not cluster:

$$\delta_k = 0$$

dark matter has no pressure of its own; it is not coupled to photons, so there is no restoring pressure force.

$$\ddot{\delta}_k + 2H\dot{\delta}_k + [\underbrace{\cancel{\frac{c_s^2 k^2}{a^2}}}_{\text{zero}} - 4\pi G\rho_0]\,\delta_k = 0$$

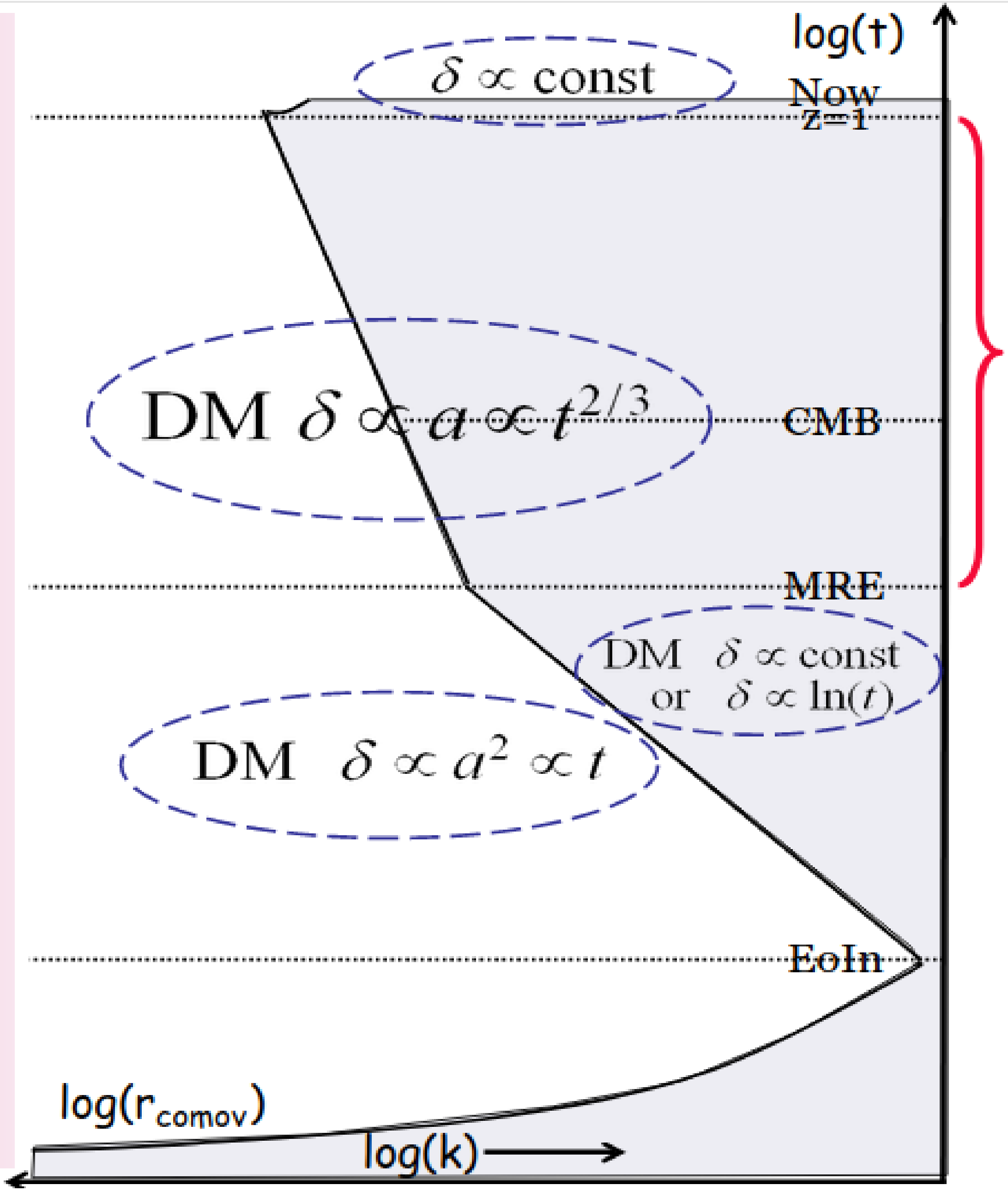
$$\ddot{\delta}_k + \tfrac{2}{t}\dot{\delta}_k = 0$$

$$\delta_k = \underbrace{A}_{\text{"growing" mode}} + \underbrace{Bt^{-1}}_{\text{decaying mode}}$$

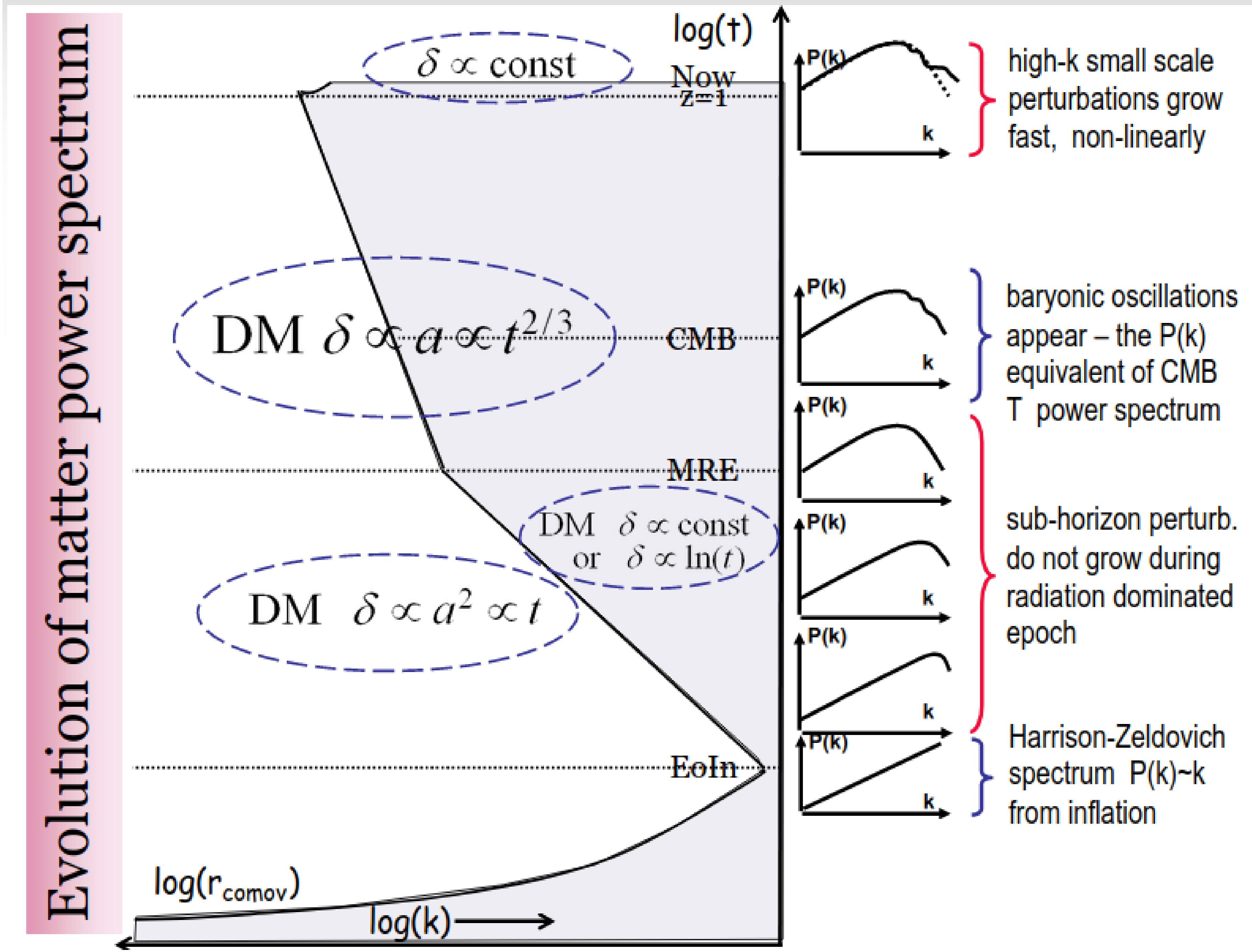
Two linearly indep. solutions: **growing** mode always comes to dominate; ignore **decaying** mode soln.

$$\boxed{\text{DM "growing" mode soln} \quad \delta_k \propto \text{const}}$$

Evolution of matter power spectrum



On sub-horizon scales
growth of structure begins and ends with matter domination


Evolution of matter power spectrum
log(t)
$\delta \propto$ const
Now
z=1
DM $\delta \propto a \propto t^{2/3}$
CMB
MRE
DM $\delta \propto$ const or $\delta \propto \ln(t)$
DM $\delta \propto a^2 \propto t$
EoIn
log(r_{comov})
log(k)
P(k)
k
high-k small scale perturbations grow fast, non-linearly
baryonic oscillations appear – the P(k) equivalent of CMB T power spectrum
sub-horizon perturb. do not grow during radiation dominated epoch
Harrison-Zeldovich spectrum P(k)~k from inflation

Growth of large scale structure

Dark Matter density maps from N-body simulations

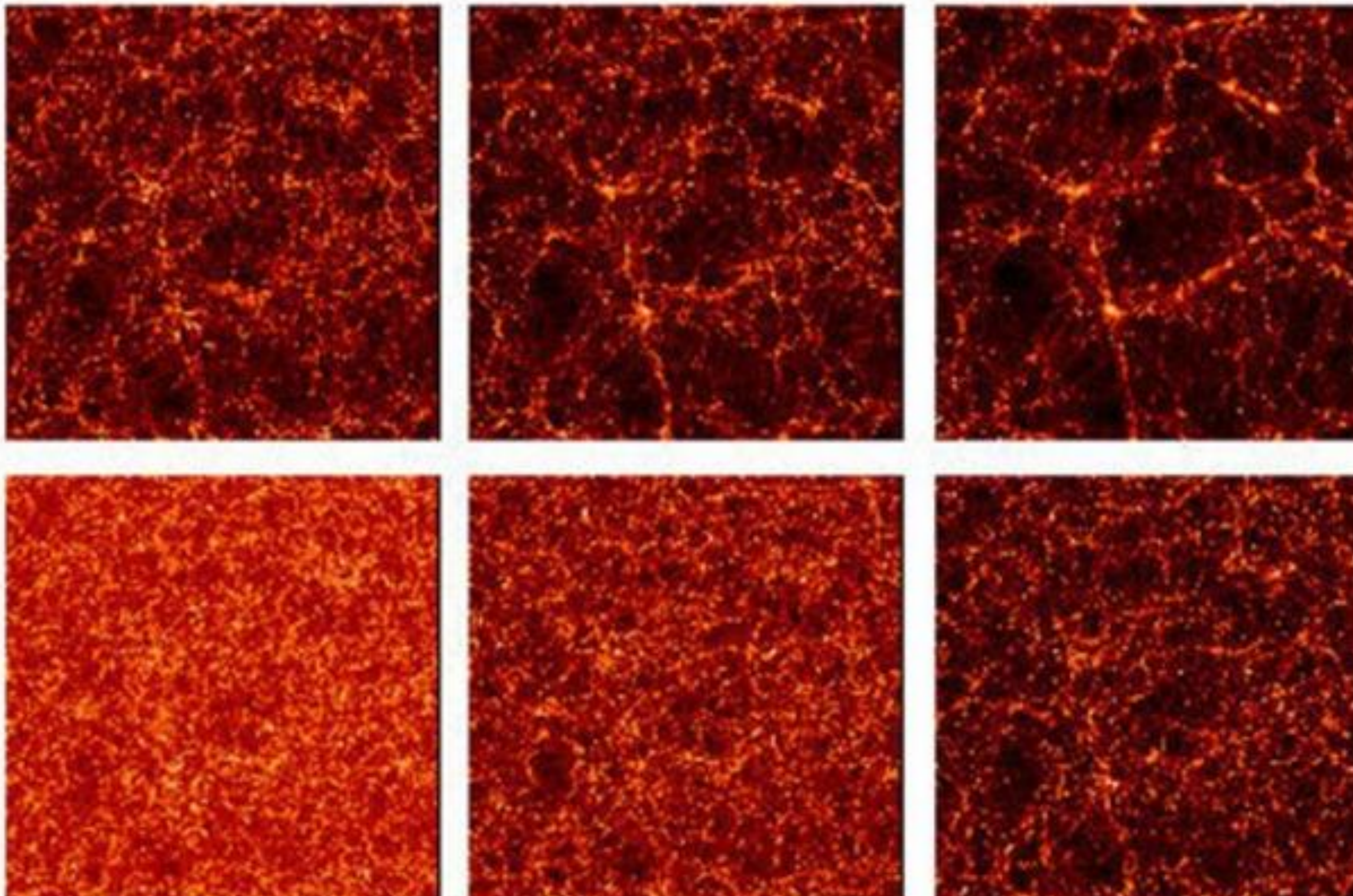
z=3 z=1 z=0

Lambda (DE)
spatially flat
Ω_{matter}=0.3

fractional overdensity ~const

Standard
spatially flat
Ω_{matter}=1.0

fractional overdensity ~1/(1+z)



← 350 Mpc →

the Virgo Collaboration (1996

Summary:
Standard Model of Cosmology

ΛCDM Paradigm + Inflation

$$\boxed{H(t)^2 + \frac{k}{a(t)^2} = \frac{8\pi G}{3}\left[\rho_{dm}(t) + \rho_b(t) + \rho_r(t)\right] + \frac{\Lambda}{3}}$$

$$\boxed{w_\Lambda \equiv \frac{p_\Lambda}{\rho_\Lambda} = -1}$$

$$\boxed{\dot{H}(t) - \frac{k}{a(t)^2} = -4\pi G\left[\rho_{dm}(t) + p_{dm}(t) + \rho_b(t) + p_b(t) + \rho_r(t) + p_r(t)\right]}$$

ΛCDM concordance model is almost perfect!

- Describes the thermal history of the Universe at the background level
- Epochs of inflation, radiation, matter, late-time acceleration

Cosmology-background

- Homogeneity and isotropy: $ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$

- Background evolution (Friedmann equations) in flat space

$$H^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_{DE}\right)$$
$$\dot{H} = -4\pi G\left(\rho_m + p_m + \rho_{DE} + p_{DE}\right),$$

(the effective DE sector can be either Λ or any possible modification)

- One must obtain a H(z) and Ωm(z) and wDE(z) in agreement with observations (SNIa, BAO, CMB shift parameter, H(z) etc)

Cosmology-perturbations

- Perturbation evolution: $\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\rm eff}\,\rho\,\delta \approx 0$ where $\delta \equiv \delta\rho/\rho$

where $G_{\rm eff}(z,k)$ is the effective Newton's constant, given by

$$\nabla^2\phi \approx 4\pi G_{\rm eff}\rho\,\delta.$$

under the scalar metric perturbation $ds^2 = -(1+2\phi)dt^2 + a^2(1-2\psi)d\vec{x}^{\,2}$

- Hence: $\delta'' + \left(\frac{(H^2)'}{2\,H^2} - \frac{1}{1+z}\right)\delta' \approx \frac{3}{2}(1+z)\frac{H_0^2}{H^2}\frac{G_{\rm eff}(z,k)}{G_N}\,\Omega_{0m}\delta$

with $f(a) = \frac{dln\delta}{dlna}$ the growth rate, with $f(a) = \Omega_{\rm m}(a)^{\gamma(a)}$ and $\Omega_{\rm m}(a) \equiv \frac{\Omega_{0m}\,a^{-3}}{H(a)^2/H_0^2}$

- One can define the observable: $\boxed{f\sigma_8(a) \equiv f(a)\cdot\sigma(a) = \frac{\sigma_8}{\delta(1)}\,a\,\delta'(a)}$

with $\sigma(a) = \sigma_8\frac{\delta(a)}{\delta 1}$ the z-dependent rms fluctuations of the linear density field within spheres of radius $R = 8h^{-1}\mathrm{Mpc}$, and σ8 its value today.

Cosmology-perturbations

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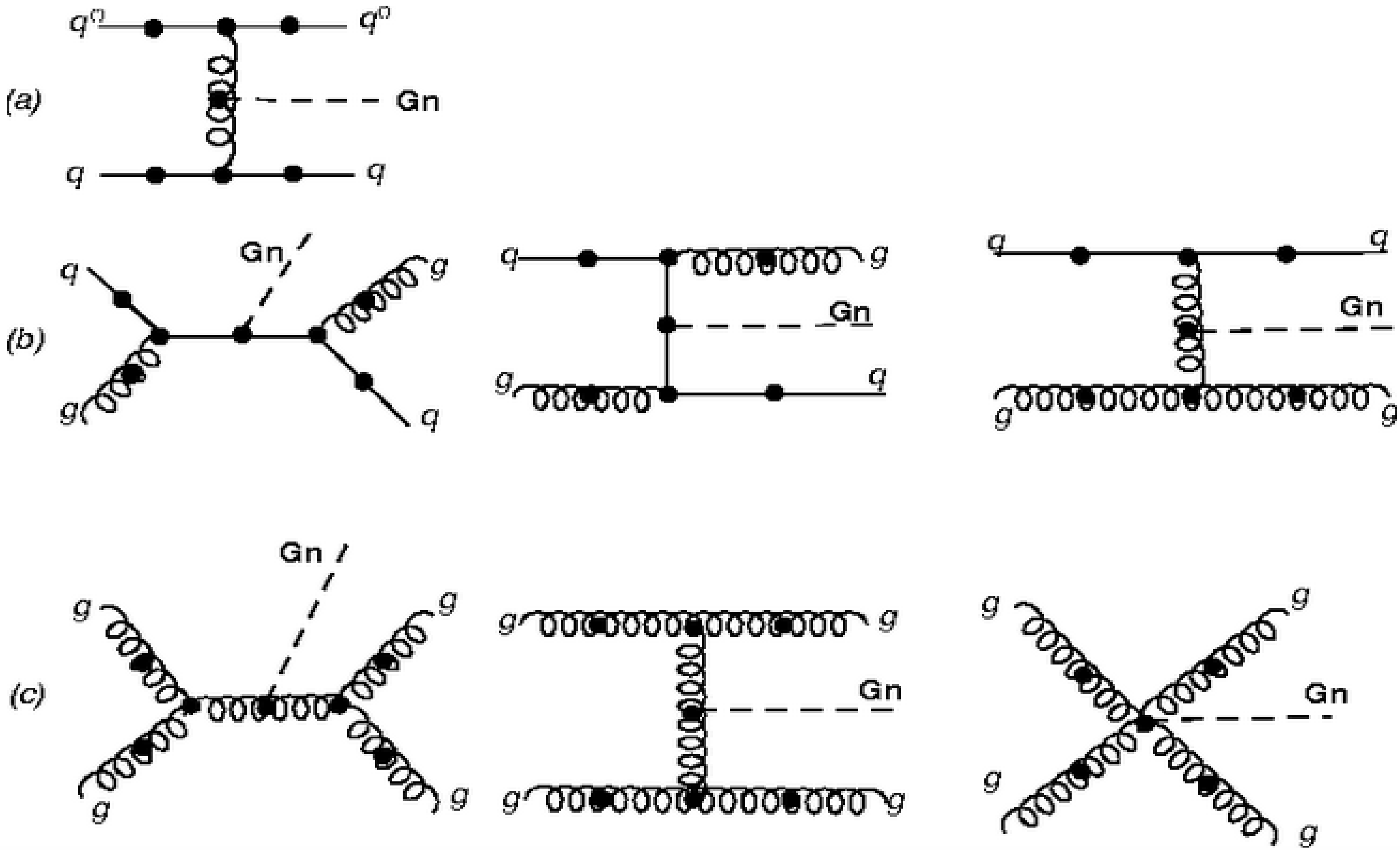
Issues of ΛCDM Paradigm

- ΛCDM is a successful cosmological model:

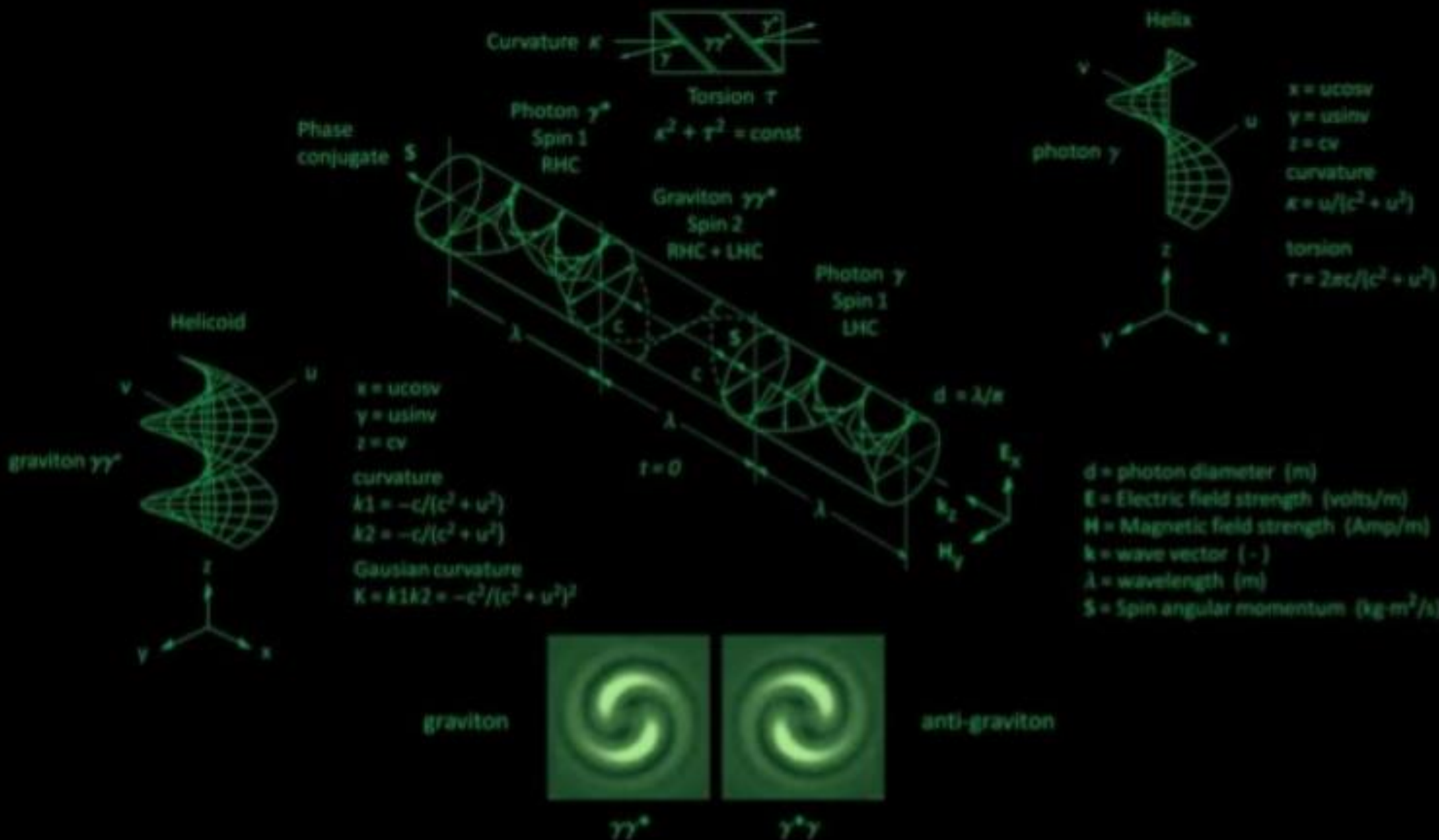
1) Describes the evolution of the universe at the background level
2) Describes the evolution of the universe at the perturbation level

- However there are open issues:
  1) General Relativity is non-renormalizable. It cannot get quantized.
  2) The cosmological-constant problem. Calculation of Λ gives a number 120 orders of magnitude larger than observed. Worst error in the ~~history of physics~~, ~~history of science~~, history
  3) How to describe primordial universe (inflation)
  4) Tensions with some data sets, e.g. H0, fσ8, AL data
  5) Missing galaxy satellites, curspy-core problems.

Can General Relativity be quantized?



Graviton



Curvature κ
Torsion τ
$\kappa^2 + \tau^2 = \text{const}$
Phase conjugate
Photon γ^*
Spin 1
RHC
Graviton $\gamma\gamma^*$
Spin 2
RHC + LHC
Photon γ
Spin 1
LHC
Helix
photon γ
x = ucosv
y = usinv
z = cv
curvature
$\kappa = u/(c^2 + u^2)$
torsion
$\tau = 2\pi c/(c^2 + u^2)$
Helicoid
graviton $\gamma\gamma^*$
x = ucosv
y = usinv
z = cv
curvature
$k1 = -c/(c^2 + u^2)$
$k2 = -c/(c^2 + u^2)$
Gausian curvature
$K = k1k2 = -c^2/(c^2 + u^2)^2$
$t = 0$
$d = \lambda/\pi$
d = photon diameter (m)
E = Electric field strength (volts/m)
H = Magnetic field strength (Amp/m)
k = wave vector (-)
λ = wavelength (m)
S = Spin angular momentum (kg·m²/s)
graviton
anti-graviton
$\gamma\gamma^*$
$\gamma^*\gamma$

Standard Model vs General Relativity Lagrangians

1

$$-\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu +$$

$$\tfrac{1}{2}ig_s^2(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu -$$

2

$$M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \tfrac{1}{2}\partial_\mu H \partial_\mu H -$$

$$\tfrac{1}{2}m_h^2 H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - M^2\phi^+\phi^- - \tfrac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \tfrac{1}{2c_w^2}M\phi^0\phi^0 - \beta_h[\tfrac{2M^2}{g^2} +$$

$$\tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu -$$

$$W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu -$$

$$W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu -$$

$$W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu +$$

$$\tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) +$$

$$g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu -$$

$$W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] -$$

$$\tfrac{1}{8}g^2\alpha_h[H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2] -$$

$$gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) -$$

$$W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \tfrac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ -$$

$$\phi^+\partial_\mu H)] + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - ig\tfrac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) +$$

$$igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\tfrac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) +$$

$$igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \tfrac{1}{4}g^2 W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] -$$

$$\tfrac{1}{4}g^2\tfrac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-] - \tfrac{1}{2}g^2\tfrac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- +$$

$$W^-_\mu\phi^+) - \tfrac{1}{2}ig^2\tfrac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- +$$

$$W^-_\mu\phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- -$$

$$g^1 s_w^2 A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j -$$

3

$$\bar{d}^\lambda_j(\gamma\partial + m_d^\lambda)d^\lambda_j + igs_w A_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)] +$$

$$\tfrac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\tfrac{4}{3}s_w^2 -$$

$$1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \tfrac{8}{3}s_w^2 - \gamma^5)d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) +$$

$$(\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1 +$$

$$\gamma^5)u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}\tfrac{m_e^\lambda}{M}[-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)] -$$

4

$$\tfrac{g}{2}\tfrac{m_e^\lambda}{M}[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}}\phi^+[-m_d^\kappa(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) +$$

$$m_u^\lambda(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}}\phi^-[m_d^\lambda(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m_u^\kappa(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 -$$

$$\gamma^5)u^\kappa_j] - \tfrac{g}{2}\tfrac{m_u^\lambda}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \tfrac{g}{2}\tfrac{m_d^\lambda}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m_u^\lambda}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) -$$

$$\tfrac{ig}{2}\tfrac{m_d^\lambda}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j) + \bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 -$$

5

$$\tfrac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_w W^+_\mu(\partial_\mu\bar{X}^0 X^- - \partial_\mu\bar{X}^+ X^0) + igs_w W^+_\mu(\partial_\mu\bar{Y}X^- -$$

$$\partial_\mu\bar{X}^+ Y) + igc_w W^-_\mu(\partial_\mu\bar{X}^- X^0 - \partial_\mu\bar{X}^0 X^+) + igs_w W^-_\mu(\partial_\mu\bar{X}^- Y -$$

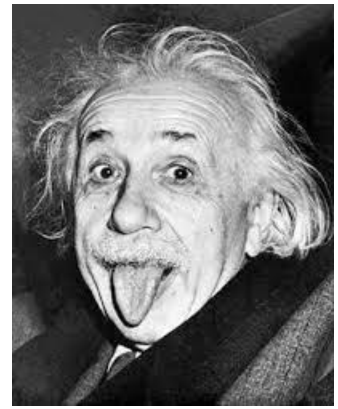
$$\partial_\mu\bar{Y}X^+) + igc_w Z^0_\mu(\partial_\mu\bar{X}^+ X^+ - \partial_\mu\bar{X}^- X^-) + igs_w A_\mu(\partial_\mu\bar{X}^+ X^+ -$$

$$\partial_\mu\bar{X}^- X^-) - \tfrac{1}{2}gM[\bar{X}^+ X^+ H + \bar{X}^- X^- H + \tfrac{1}{c_w^2}\bar{X}^0 X^0 H] +$$

$$\tfrac{1-2c_w^2}{2c_w}igM[\bar{X}^+ X^0\phi^+ - \bar{X}^- X^0\phi^-] + \tfrac{1}{2c_w}igM[\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-] +$$

$$igMs_w[\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-] + \tfrac{1}{2}igM[\bar{X}^+ X^+\phi^0 - \bar{X}^- X^-\phi^0]$$

$$S = -\frac{1}{16\pi G}\int\sqrt{-g}\,(R(g) + 2\Lambda)\,\mathrm{d}^4x$$



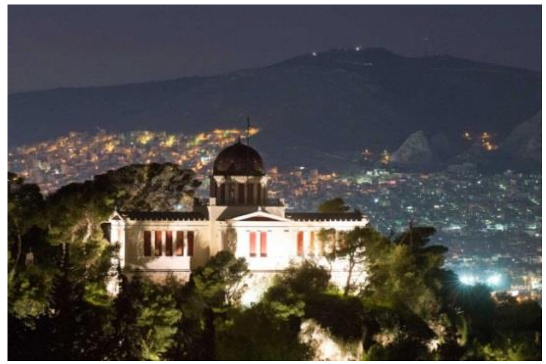
COSMOLOGICAL CONSTANT PROBLEM

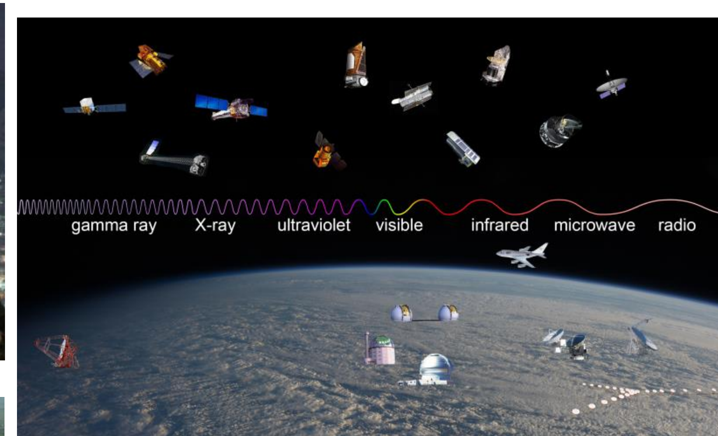
$$E_n \sim (n + 1/2)h\omega(k)$$

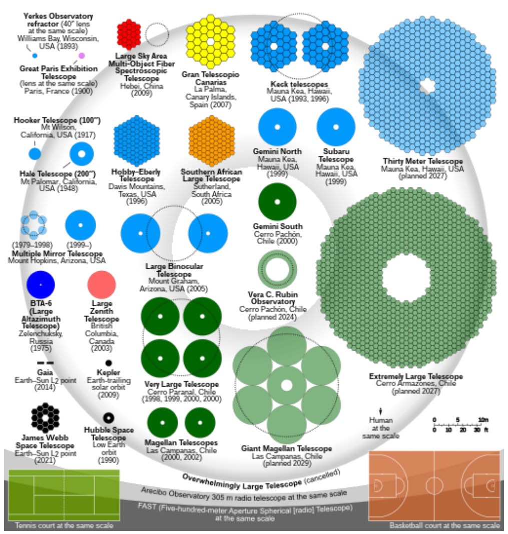
$$\rho_\Lambda(th) \sim M_p^4$$

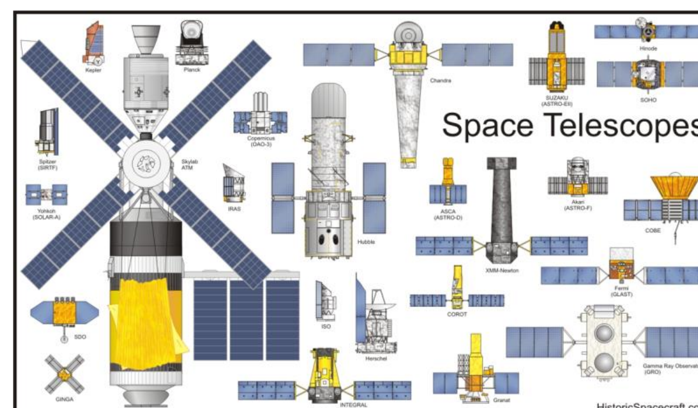
$$\rho_\Lambda^0 \sim 10^{-120}\rho_\Lambda^{th}$$

Astrophysics and Cosmology in the 21st century



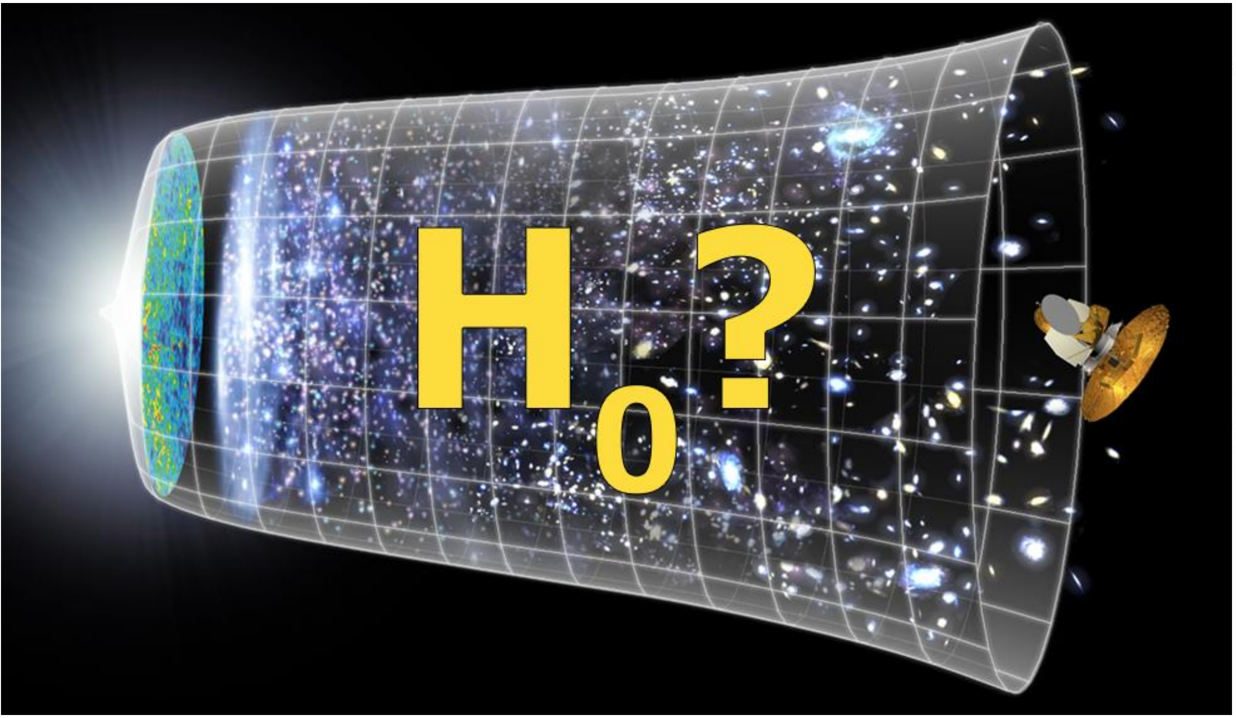


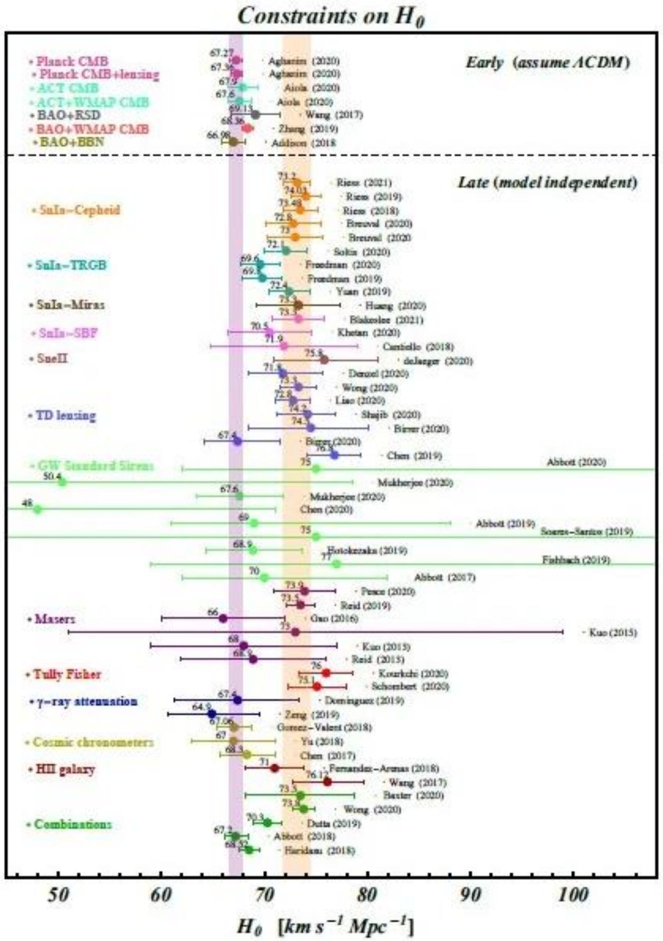




H0 tension

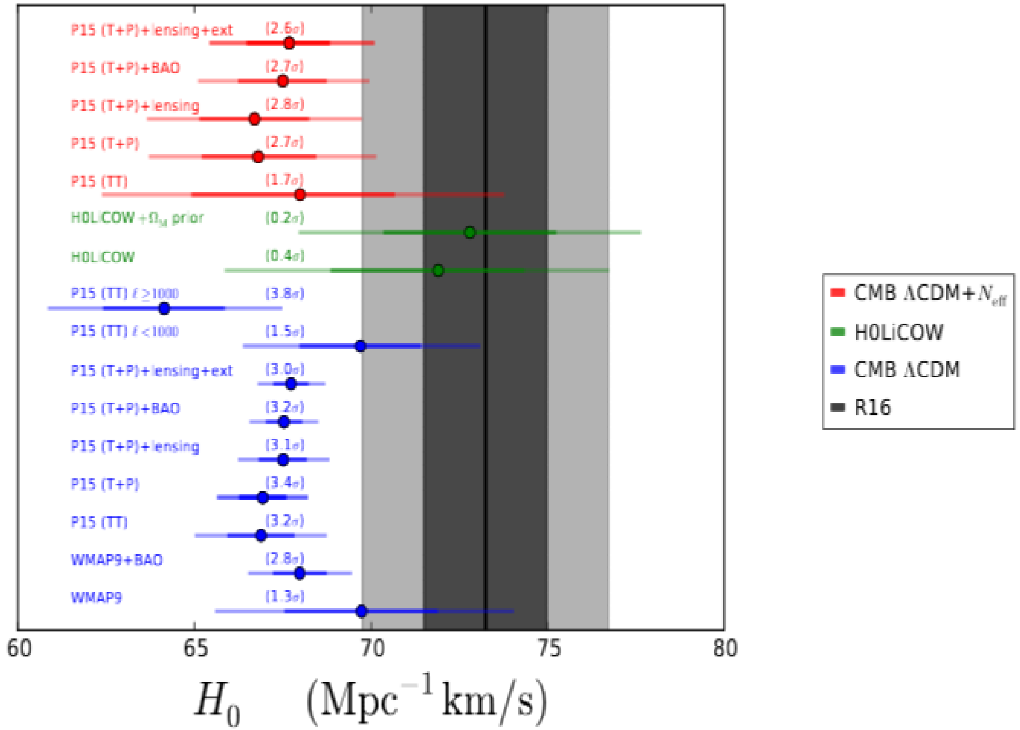
- The Universe expands faster than expected!



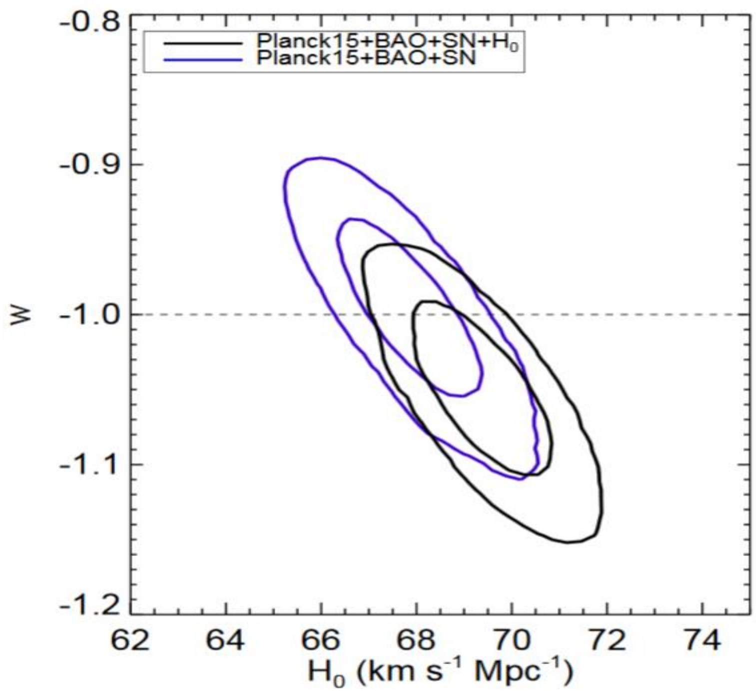


Tension1 – H0

- Tension between the data (direct measurements) and Planck/ΛCDM (indirect measurements). The data indicate a lack of "gravitational power".



[Bernal, Verde, Riess, JCAP1610]

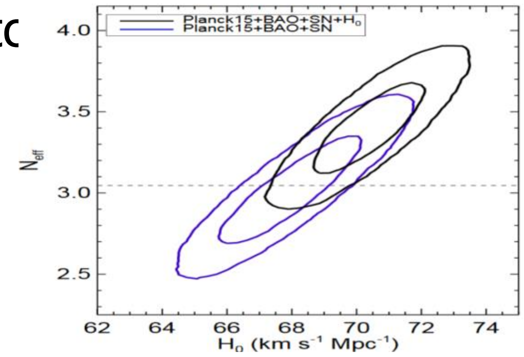


[Riess et al, Astrophys.J 826]

Tension1 – H0

- Tension between the data (direct measurements) and Planck/ΛCDM (indirect measurements). The data indicate a lack of "gravitational power".
- This tension could be due to systematics.

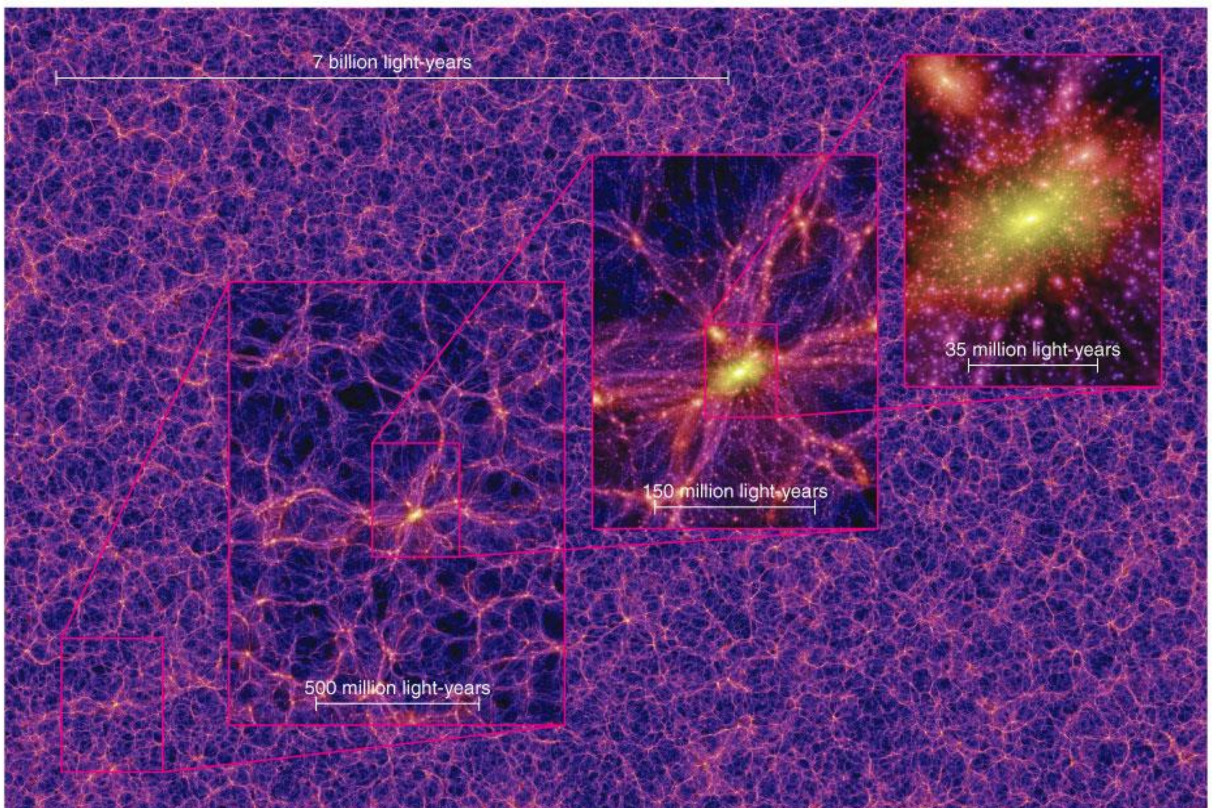
- If not systematics then we may need changes in ΛCDM in early or late time behavior.
- Higher number of effective relativistic species, dynamical dark energy, non-zero curvature, etc

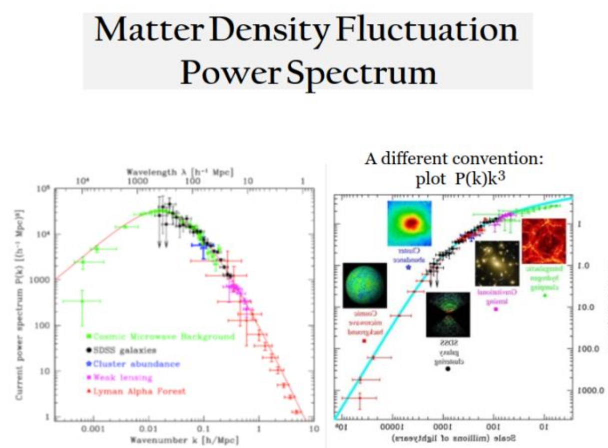


- Or Modified Gravity.

S8 Tension

- "Less" Matter clustering than expected!

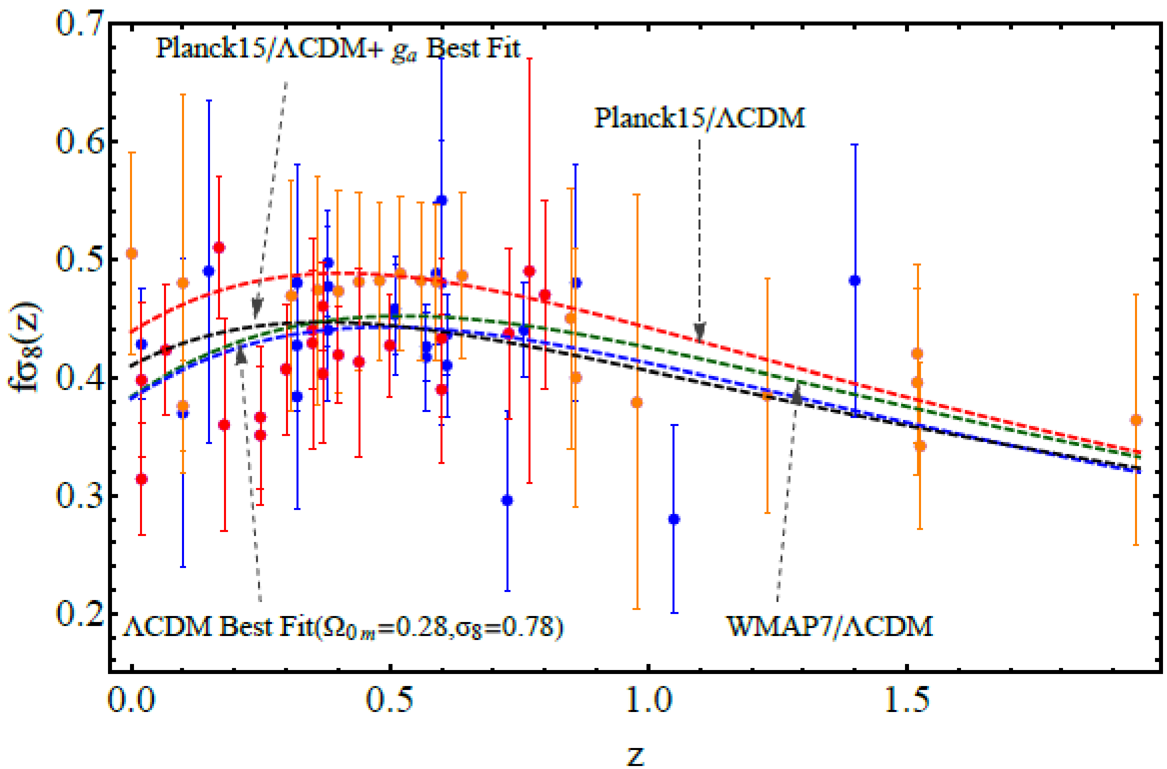




Tension2 – fσ8

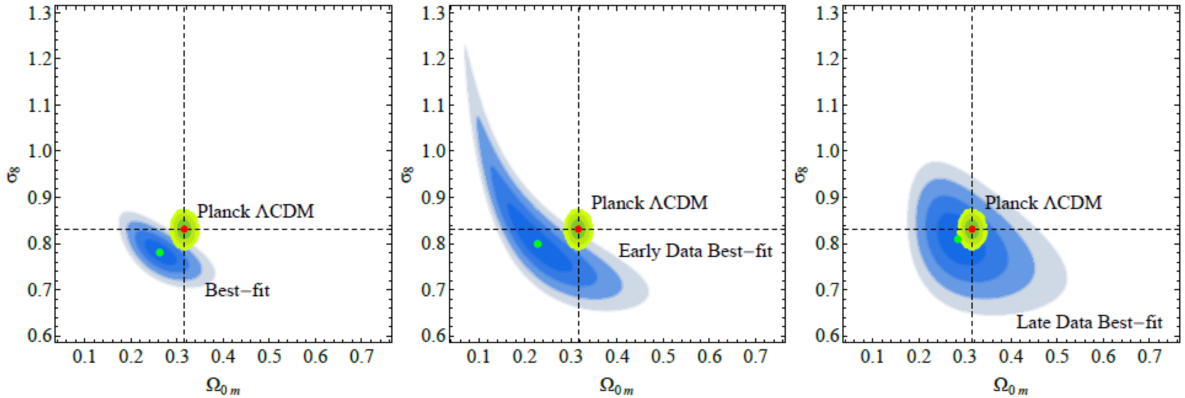
- Tension between the data and Planck/ΛCDM. The data indicate a lack of "gravitational power" in structures on intermediate-small cosmological scales.

Parameter	Planck15/ΛCDM [12]	WMAP7/ΛCDM [45]
$\Omega_b h^2$	0.02225 ± 0.00016	0.02258 ± 0.00057
$\Omega_c h^2$	0.1198 ± 0.0015	0.1109 ± 0.0056
n_s	0.9645 ± 0.0049	0.963 ± 0.014
H_0	67.27 ± 0.66	71.0 ± 2.5
Ω_{0m}	0.3156 ± 0.0091	0.266 ± 0.025
w	-1	-1
σ_8	0.831 ± 0.013	0.801 ± 0.030



Tension2 – fσ8

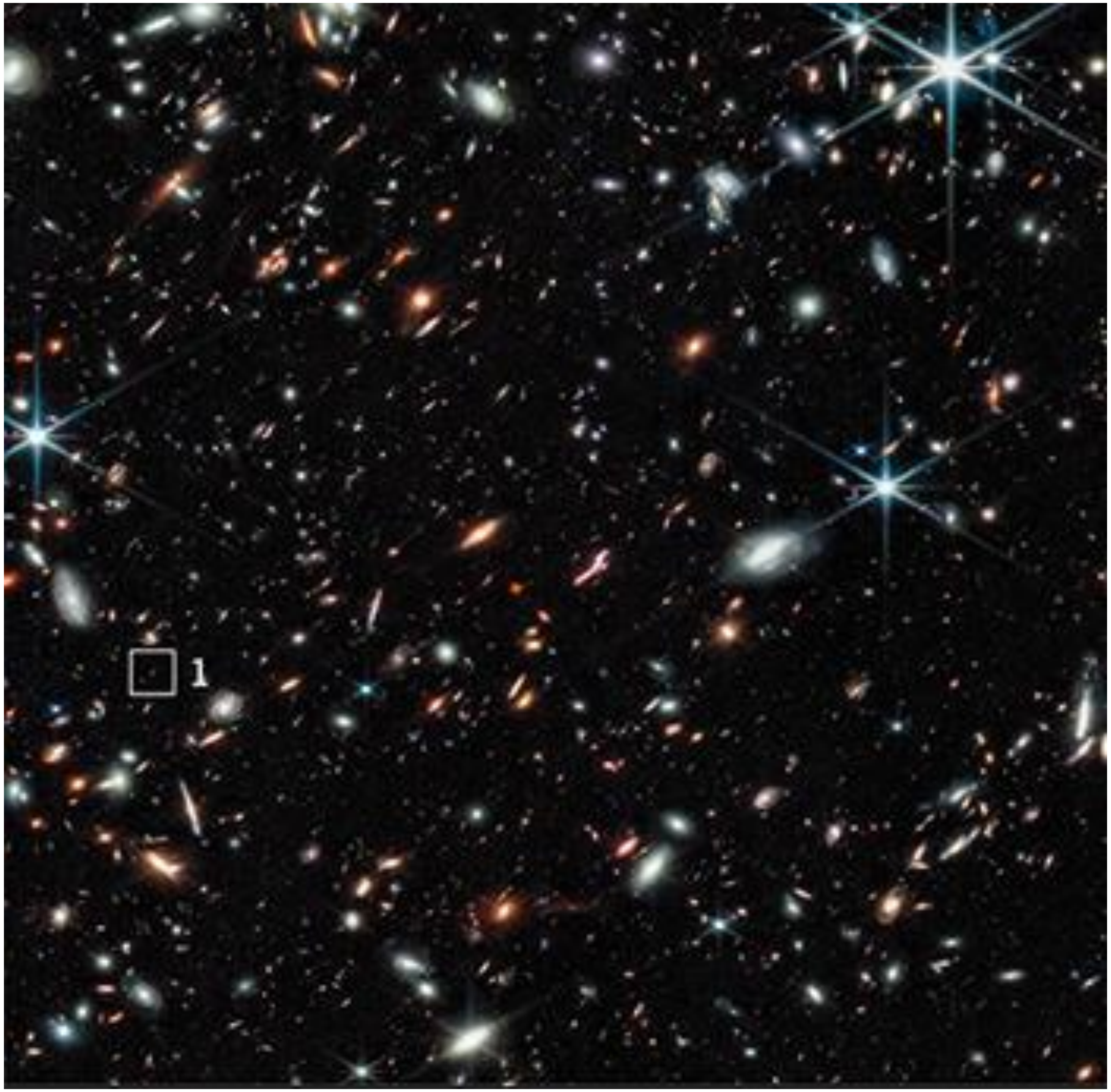
- Tension between the data and Planck/ΛCDM.
- This tension could be due to systematics. E.g:



[Kazantzidis, Perivolaropoulos, PRD97]

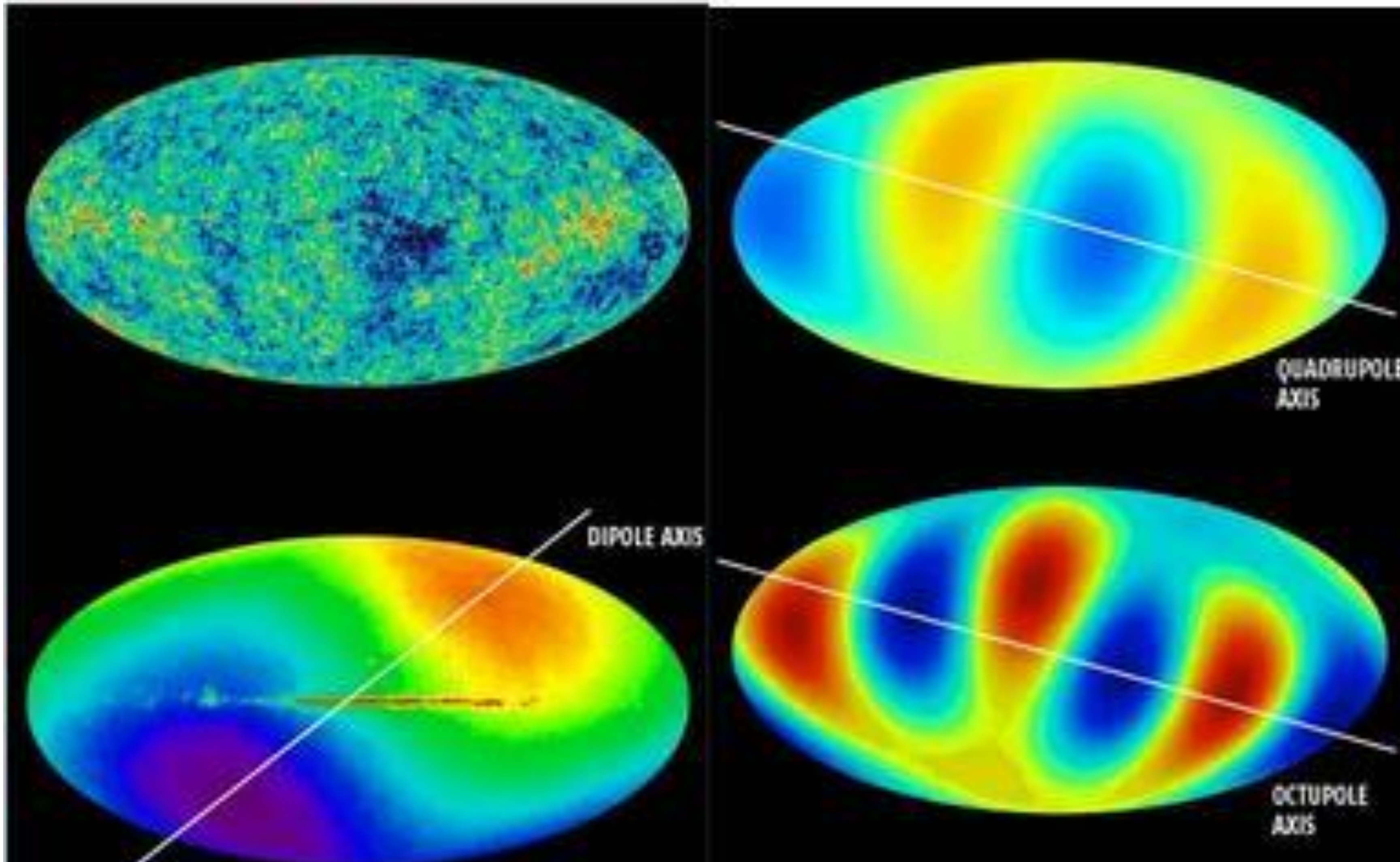
- If not systematics, the data indicate a lack of "gravitational power" in structures on intermediate-small cosmological scales (expressed as smaller Ωm at z<0.6, or smaller σ8, or wDE<-1).
- It could be reconciled by a mechanism that reduces the rate of clustering between recombination and today: Hot Dark Matter, Dark Matter that clusters differently at small scales, or Modified Gravity.

Too many galaxies too early!

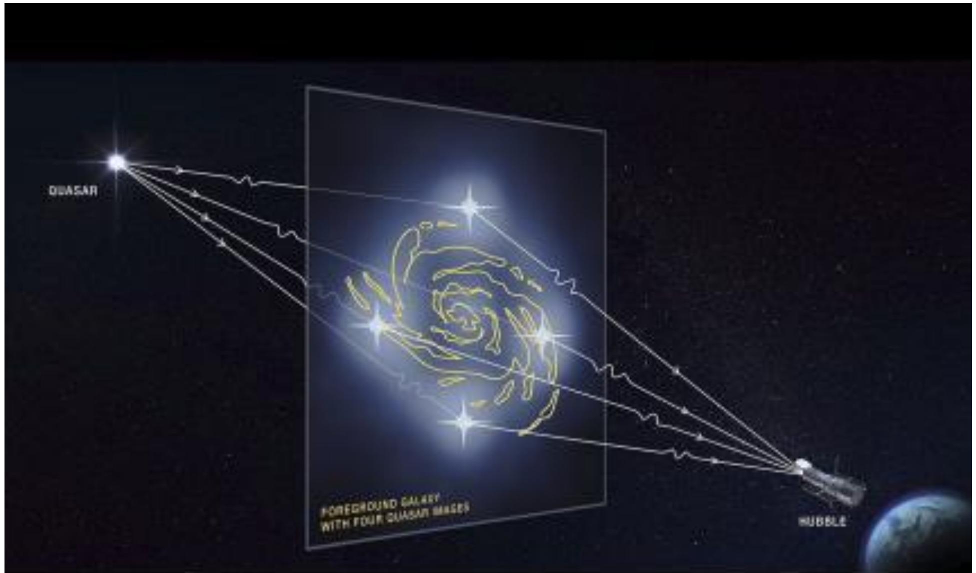


James Webb space telescope

Cosmic dipole tension!



The lensing anomaly




PBS
THE NEW CRISIS
IN COSMOLOGY

History of Major Shifts of Cosmological Models

Aristarchus of Samos (c. 310-230 BCE)

- **Geocentric (Ptolemy) to Heliocentric (Copernicus, 16th-17th c.):**
  - Retrograde motion (Copernicus)
  - Phases of Venus, Moons of Jupiter (Galileo, 1610)
- **Heliocentric to Infinite Universe (18th-19th c.):**
  - Improved telescopes, Uranus discovery (Herschel, 1781)
  - Stellar parallax (Bessel, 1838)
- **Infinite to Static Universe (Einstein, early 20th c.):**
  - Nebulae spectroscopy (Slipher)
  - Stellar distances (Leavitt, Hertzsprung)
- **Static to Expanding Universe (Lemaître, Hubble, 1920s-30s):**
  - Galactic redshift (Slipher, 1912-14)
  - Hubble's law (Hubble, 1929)
- **Expanding Universe to Inflationary Big Bang (1960s-80s):**
  - CMB (Penzias & Wilson, 1964)
  - Light element abundance (Alpher, Herman)
  - Inflation theory (Guth, Linde, 1980s)
- **Introduction of Dark Matter (1970s-80s):**
  - Galaxy rotation curves (Rubin, 1970s)
  - Gravitational lensing (Walsh et al., 1979)
  - Galaxy clusters (Zwicky, 1930s)
- **Lambda-CDM (late 1990s-present):**
  - Supernova observations (Perlmutter, Schmidt, Riess, 1998-99)
  - CMB (WMAP, Planck), BAO (SDSS, 2005)
- **Potential Future Shift (2020s-?):**
  - Hubble tension (Riess et al. vs Planck Collaboration)
  - S8 (growth rate) tension (KiDS, DES, Planck collaborations)
  - Cosmic dipoles tension (Various teams)
  - CMB anomalies (Planck Collaboration)
  - ISW (Integrated Sachs-Wolfe) tension
  - Lithium problem (Primordial Nucleosynthesis)

ELSEVIER

New Astronomy Reviews

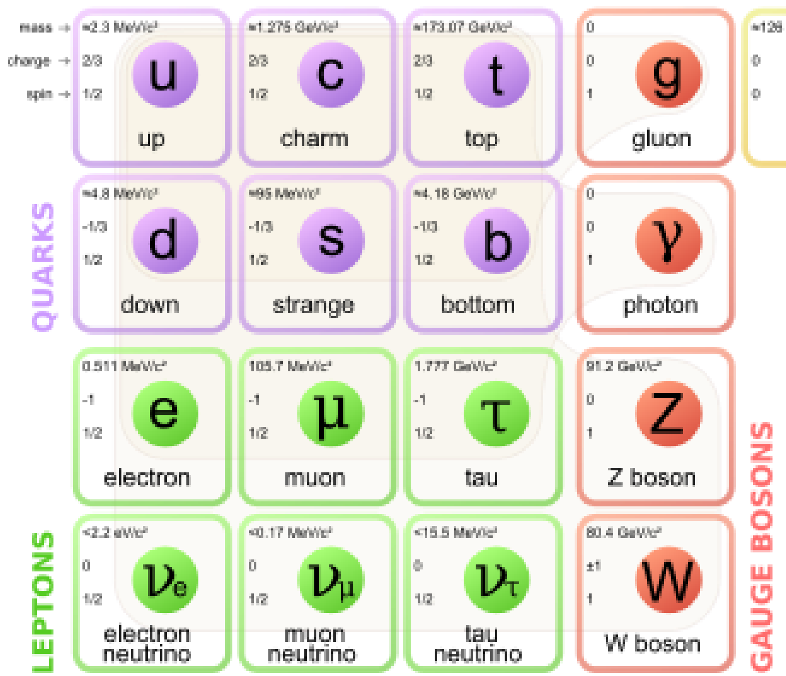
Volume 95, December 2022, 101659

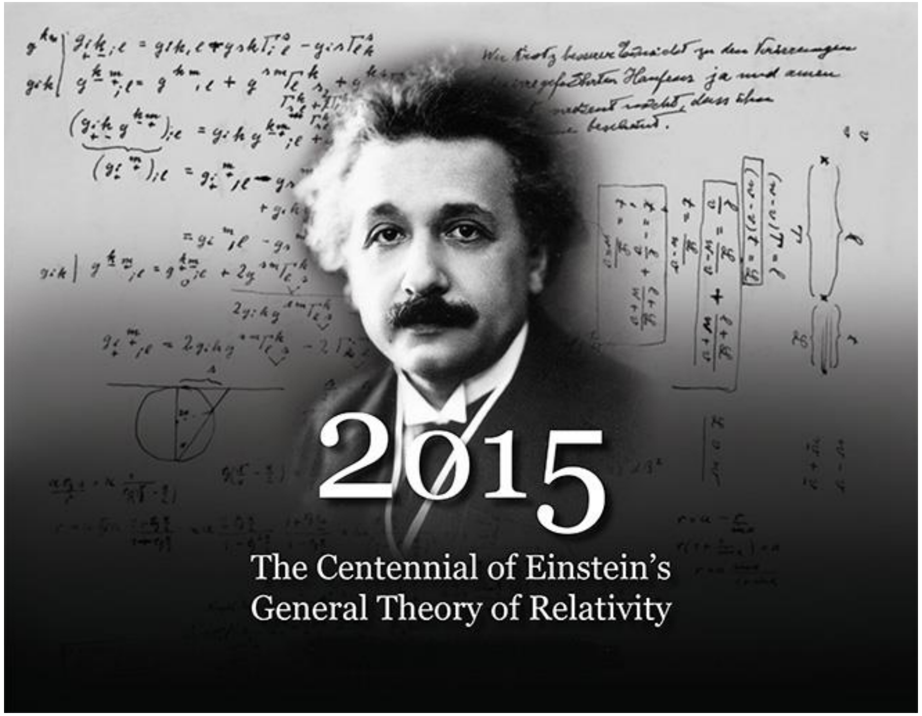
Challenges for ΛCDM: An update

L. Perivolaropoulos, F. Skara

Knowledge of Physics

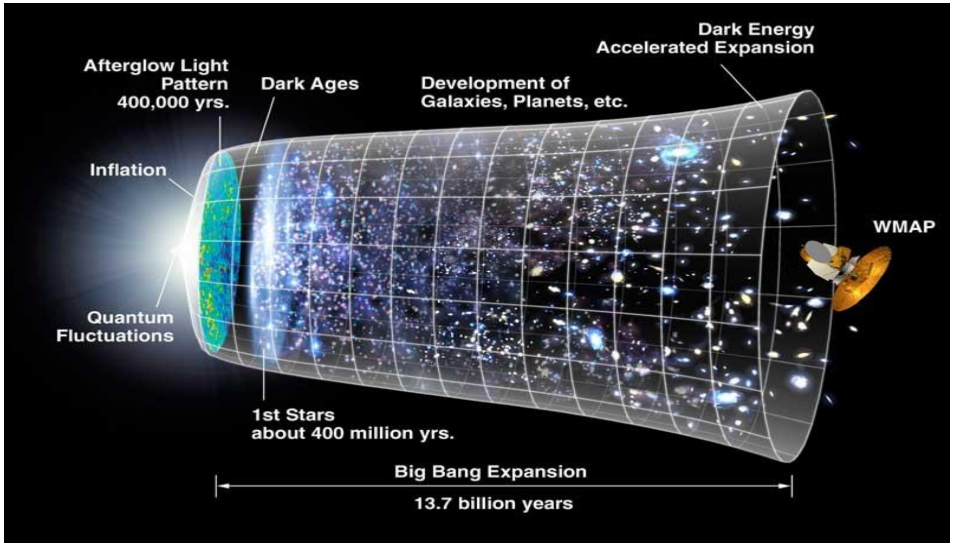
Knowledge of Physics: Standard Model + General Relativity

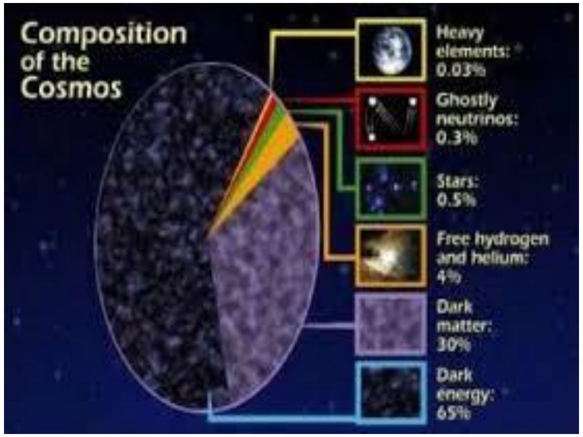




Modified/new knowledge of physics

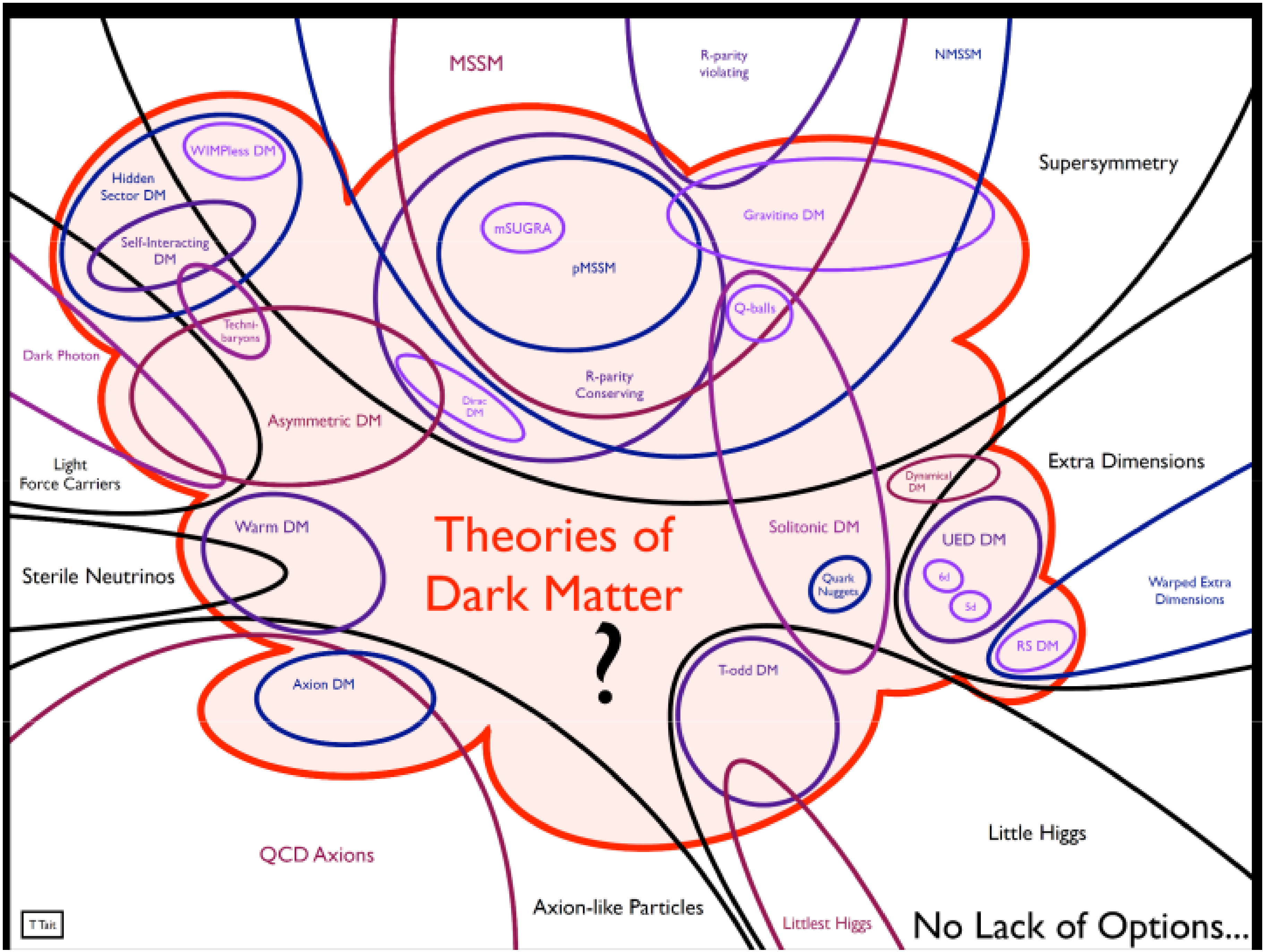
So can our knowledge of Physics describes all these?





Most probably, no!

We definitely need new physics for Inflation and Dark matter. Maybe for dark energy.



MSSM
R-parity
violating
NMSSM
Supersymmetry
WIMPless DM
Hidden
Sector DM
Self-Interacting
DM
Techni-
baryons
mSUGRA
pMSSM
Gravitino DM
Q-balls
Dark Photon
R-parity
Conserving
Dirac
DM
Asymmetric DM
Light
Force Carriers
Extra Dimensions
Dynamical
DM
Warm DM
Theories of
Dark Matter
?
Solitonic DM
UED DM
6d
5d
Quark
Nuggets
Warped Extra
Dimensions
Sterile Neutrinos
RS DM
Axion DM
T-odd DM
Little Higgs
QCD Axions
Axion-like Particles
Littlest Higgs
No Lack of Options...
T Tait

Why Modified Gravity?

We need to modify something:

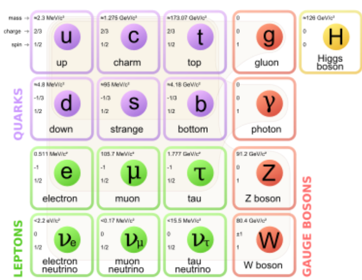
The universe content

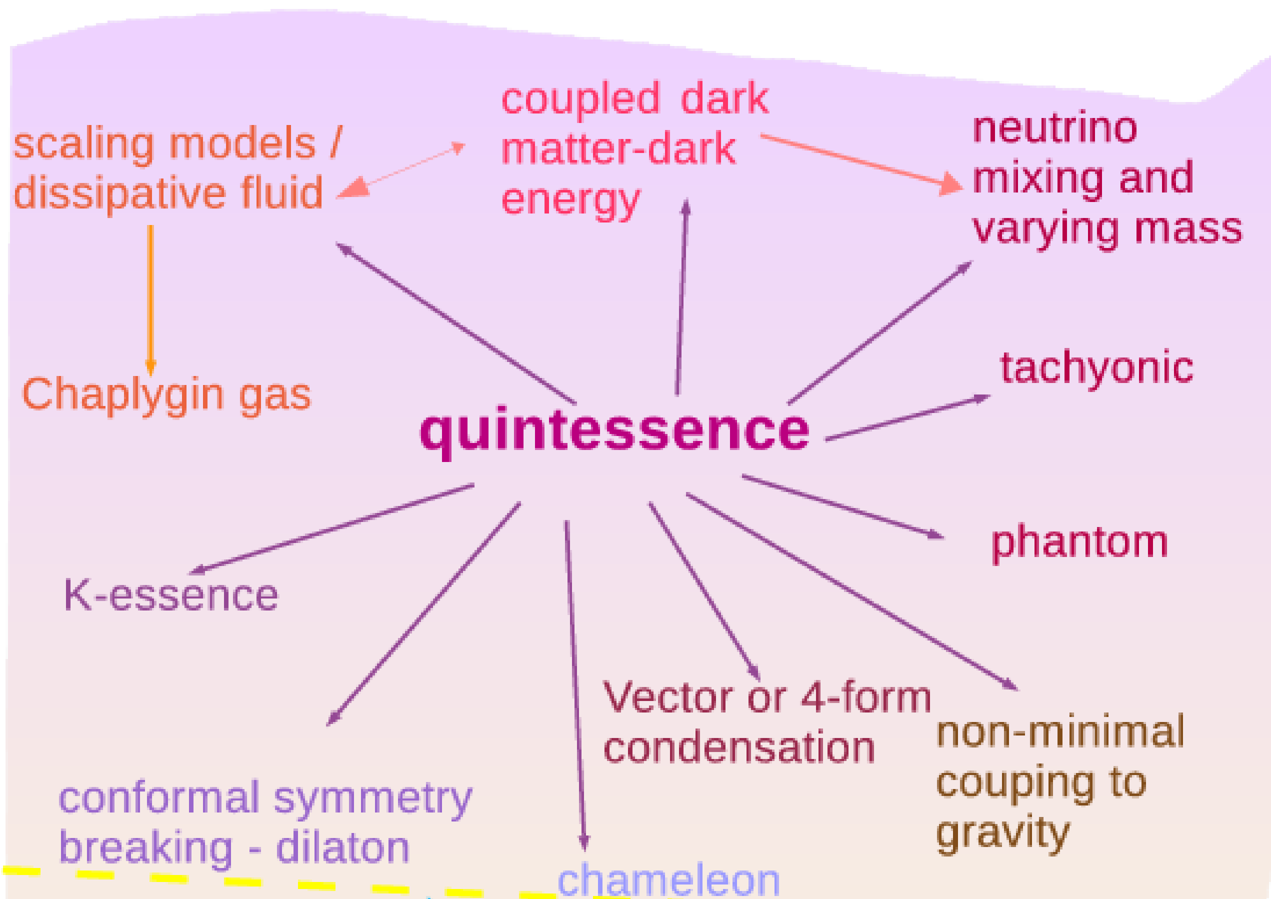
or

The theory of Gravity

Dark Energy-Inflation

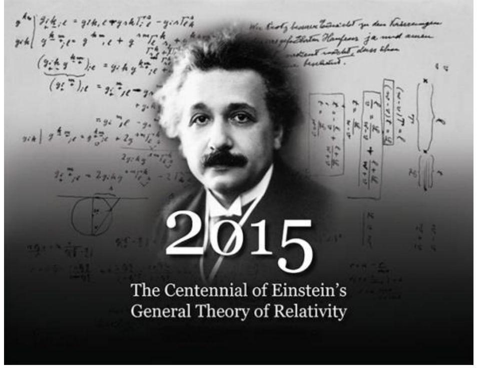
- Add a scalar field φ in the Universe content





General Relativity

- Einstein 1915: **General Relativity**:



energy-momentum source of spacetime Curvature

$$\boxed{S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R-2\Lambda] + \int d^4x\, L_m\left(g_{\mu\nu},\psi\right)}$$

$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 8\pi G\, T_{\mu\nu}$$

with $\quad T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g_{\mu\nu}}$

Standard Model vs General Relativity Lagrangians

1

$$-\frac{1}{2}\partial_\nu g^a_\mu\partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu +$$
$$\frac{1}{2}ig_s^2(\bar q^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu + \bar G^a\partial^2G^a + g_s f^{abc}\partial_\mu\bar G^a G^b g^c_\mu - \partial_\nu W^+_\mu\partial_\nu W^-_\mu -$$

2

$$M^2W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu\partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu\partial_\mu A_\nu - \frac{1}{2}\partial_\mu H\partial_\mu H -$$
$$\frac{1}{2}m_h^2H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - M^2\phi^+\phi^- - \frac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \frac{1}{2c_w^2}M\phi^0\phi^0 - \beta_h[\frac{2M^2}{g^2} +$$
$$\frac{2M}{g}H + \frac{1}{2}(H^2+\phi^0\phi^0+2\phi^+\phi^-)] + \frac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu -$$
$$W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu -$$
$$W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu -$$
$$W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - \frac{1}{2}g^2W^+_\mu W^-_\mu W^+_\nu W^-_\nu +$$
$$\frac{1}{2}g^2W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) +$$
$$g^2s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2s_wc_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu -$$
$$W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] -$$
$$\frac{1}{8}g^2\alpha_h[H^4+(\phi^0)^4+4(\phi^+\phi^-)^2+4(\phi^0)^2\phi^+\phi^-+4H^2\phi^+\phi^-+2(\phi^0)^2H^2] -$$
$$gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \frac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) -$$
$$W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \frac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ -$$
$$\phi^+\partial_\mu H)] + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) +$$
$$igs_wMA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) +$$
$$igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4}g^2W^+_\mu W^-_\mu[H^2+(\phi^0)^2+2\phi^+\phi^-] -$$
$$\frac{1}{4}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2+(\phi^0)^2+2(2s_w^2-1)^2\phi^+\phi^-] - \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) - \frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) + \frac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2-1)Z^0_\mu A_\mu\phi^+\phi^- -$$
$$g^1 s_w^2A_\mu A_\mu\phi^+\phi^- - \bar e^\lambda(\gamma\partial+m_e^\lambda)e^\lambda - \bar\nu^\lambda\gamma\partial\nu^\lambda - \bar u^\lambda_j(\gamma\partial+m_u^\lambda)u^\lambda_j -$$

3

$$\bar d^\lambda_j(\gamma\partial+m_d^\lambda)d^\lambda_j + igs_wA_\mu[-(\bar e^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar u^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar d^\lambda_j\gamma^\mu d^\lambda_j)] +$$
$$\frac{ig}{4c_w}Z^0_\mu[(\bar\nu^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar e^\lambda\gamma^\mu(4s_w^2-1-\gamma^5)e^\lambda) + (\bar u^\lambda_j\gamma^\mu(\frac{4}{3}s_w^2 -$$
$$1-\gamma^5)u^\lambda_j) + (\bar d^\lambda_j\gamma^\mu(1-\frac{8}{3}s_w^2-\gamma^5)d^\lambda_j)] + \frac{ig}{2\sqrt{2}}W^+_\mu[(\bar\nu^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) +$$
$$(\bar u^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \frac{ig}{2\sqrt{2}}W^-_\mu[(\bar e^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar d^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1+$$
$$\gamma^5)u^\lambda_j)] + \frac{ig}{2\sqrt{2}}\frac{m_e^\lambda}{M}[-\phi^+(\bar\nu^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar e^\lambda(1+\gamma^5)\nu^\lambda)] -$$

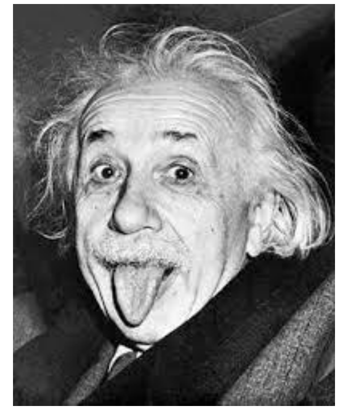
4

$$\frac{g}{2}\frac{m_e^\lambda}{M}[H(\bar e^\lambda e^\lambda) + i\phi^0(\bar e^\lambda\gamma^5e^\lambda)] + \frac{ig}{2M\sqrt{2}}\phi^+[-m_d^\kappa(\bar u^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) +$$
$$m_u^\lambda(\bar u^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j] + \frac{ig}{2M\sqrt{2}}\phi^-[m_d^\lambda(\bar d^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m_u^\kappa(\bar d^\lambda_j C^\dagger_{\lambda\kappa}(1-$$
$$\gamma^5)u^\kappa_j] - \frac{g}{2}\frac{m_u^\lambda}{M}H(\bar u^\lambda_j u^\lambda_j) - \frac{g}{2}\frac{m_d^\lambda}{M}H(\bar d^\lambda_j d^\lambda_j) + \frac{ig}{2}\frac{m_u^\lambda}{M}\phi^0(\bar u^\lambda_j\gamma^5u^\lambda_j) -$$
$$\frac{ig}{2}\frac{m_d^\lambda}{M}\phi^0(\bar d^\lambda_j\gamma^5d^\lambda_j) + \bar X^+(\partial^2-M^2)X^+ + \bar X^-(\partial^2-M^2)X^- + \bar X^0(\partial^2 -$$

5

$$\frac{M^2}{c_w^2})X^0 + \bar Y\partial^2Y + igc_wW^+_\mu(\partial_\mu\bar X^0X^- - \partial_\mu\bar X^+X^0) + igs_wW^+_\mu(\partial_\mu\bar YX^- -$$
$$\partial_\mu\bar X^+Y) + igc_wW^-_\mu(\partial_\mu\bar X^-X^0 - \partial_\mu\bar X^0X^+) + igs_wW^-_\mu(\partial_\mu\bar X^-Y -$$
$$\partial_\mu\bar YX^+) + igc_wZ^0_\mu(\partial_\mu\bar X^+X^+ - \partial_\mu\bar X^-X^-) + igs_wA_\mu(\partial_\mu\bar X^+X^+ -$$
$$\partial_\mu\bar X^-X^-) - \frac{1}{2}gM[\bar X^+X^+H + \bar X^-X^-H + \frac{1}{c_w^2}\bar X^0X^0H] +$$
$$\frac{1-2c_w^2}{2c_w}igM[\bar X^+X^0\phi^+ - \bar X^-X^0\phi^-] + \frac{1}{2c_w}igM[\bar X^0X^-\phi^+ - \bar X^0X^+\phi^-] +$$
$$igMs_w[\bar X^0X^-\phi^+ - \bar X^0X^+\phi^-] + \frac{1}{2}igM[\bar X^+X^+\phi^0 - \bar X^-X^-\phi^0]$$

$$S = -\frac{1}{16\pi G}\int\sqrt{-g}(R(g)+2\Lambda)\,\mathrm{d}^4x$$



Lovelock's Theorem

2.4.1. Lovelock's theorem

Lovelock's theorem [831, 832] limits the theories that one can construct from the metric tensor alone. To enunciate this theorem, let us begin by assuming that the metric tensor is the only field involved in the gravitational action. If the action can be written in terms of the metric tensor $g_{\mu\nu}$ alone, then we can write

$$S = \int d^4x \mathcal{L}(g_{\mu\nu}). \tag{56}$$

If this action contains up to second derivatives of $g_{\mu\nu}$, then extremising it with respect to the metric gives the Euler-Lagrange expression

$$E^{\mu\nu}[\mathcal{L}] = \frac{d}{dx^\rho}\left[\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\rho}} - \frac{d}{dx^\lambda}\left(\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\rho\lambda}}\right)\right] - \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}, \tag{57}$$

and the Euler-Lagrange equation is $E^{\mu\nu}(\mathcal{L}) = 0$. Lovelock's theorem can then be stated as the following:

Theorem 2.1. (Lovelock's Theorem)
The only possible second-order Euler-Lagrange expression obtainable in a four dimensional space from a scalar density of the form $\mathcal{L} = \mathcal{L}(g_{\mu\nu})$ is

$$E^{\mu\nu} = \alpha\sqrt{-g}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right] + \lambda\sqrt{-g}g^{\mu\nu}, \tag{58}$$

Lovelock's Theorem

where α and λ are constants, and $R_{\mu\nu}$ and R are the Ricci tensor and scalar curvature, respectively.

This powerful theorem means that if we try to create any gravitational theory in a four-dimensional Riemannian space from an action principle involving the metric tensor and its derivatives only, then the only field equations that are second order or less are Einstein's equations and/or a cosmological constant. This does not, however, imply that the Einstein-Hilbert action is the only action constructed from $g_{\mu\nu}$ that results in the Einstein equations. In fact, in four dimensions or less one finds that the most general such action is

$$\mathcal{L} = \alpha\sqrt{-g}R - 2\lambda\sqrt{-g} + \beta\epsilon^{\mu\nu\rho\lambda}R^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta\rho\lambda} + \gamma\sqrt{-g}\left(R^2 - 4R^{\mu}{}_{\nu}R^{\nu}{}_{\mu} + R^{\mu\nu}{}_{\rho\lambda}R^{\rho\lambda}{}_{\mu\nu}\right),$$

where β and γ are also constants. The third and fourth terms in this expression do not, however, contribute to the Euler-Lagrange equations as

$$E^{\mu\nu}\left[\epsilon^{\alpha\beta\rho\lambda}R^{\gamma\delta}{}_{\alpha\beta}R_{\gamma\delta\rho\lambda}\right] = 0 \quad (59)$$

$$E^{\mu\nu}\left[\sqrt{-g}\left(R^2 - 4R^{\alpha}{}_{\beta}R^{\beta}{}_{\alpha} + R^{\alpha\beta}{}_{\rho\lambda}R^{\rho\lambda}{}_{\alpha\beta}\right)\right] = 0, \quad (60)$$

where the action of $E^{\mu\nu}$ on any function X is defined as in Eq. (57). The first of these equations is valid in any number of dimensions, and the second is valid in four dimensions only.

Lovelock's Theorem

Lovelock's theorem means that to construct metric theories of gravity with field equations that differ from those of General Relativity we must do one (or more) of the following:

- Consider other fields, beyond (or rather than) the metric tensor.
- Accept higher than second derivatives of the metric in the field equations.
- Work in a space with dimensionality different from four.
- Give up on either rank (2,0) tensor field equations, symmetry of the field equations under exchange of indices, or divergence-free field equations.
- Give up locality.

General Relativity Assumptions and Considerations

$$\boxed{S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R-2\Lambda\right]+\int d^4x\; L_m\left(g_{\mu\nu},\psi\right)}$$

- Diffeomorphism invariance

General Relativity
Assumptions and Considerations

$$\boxed{S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R-2\Lambda\right]+\int d^4x\, L_m\left(g_{\mu\nu},\psi\right)}$$

- Diffeomorphism invariance
- Spacetime dimensionality=4

General Relativity
Assumptions and Considerations

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R-2\Lambda\right]+\int d^4x\, L_m\left(g_{\mu\nu},\psi\right)$$

- Diffeomorphism invariance
- Spacetime dimensionality=4
- Geometry=Curvature (connection=Levi Civita)
- Linear in Ricci scalar

General Relativity Assumptions and Considerations

$$\boxed{S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - 2\Lambda\right] + \int d^4x\, L_m\left(g_{\mu\nu}, \psi\right)}$$

- Diffeomorphism invariance
- Spacetime dimensionality=4
- Geometry=Curvature (connection=Levi Civita)
- Linear in Ricci scalar
- Metric compatibility (zero non-metricity)

General Relativity
Assumptions and Considerations

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - 2\Lambda\right] + \int d^4x\, L_m\left(g_{\mu\nu},\psi\right)$$

- Diffeomorphism invariance
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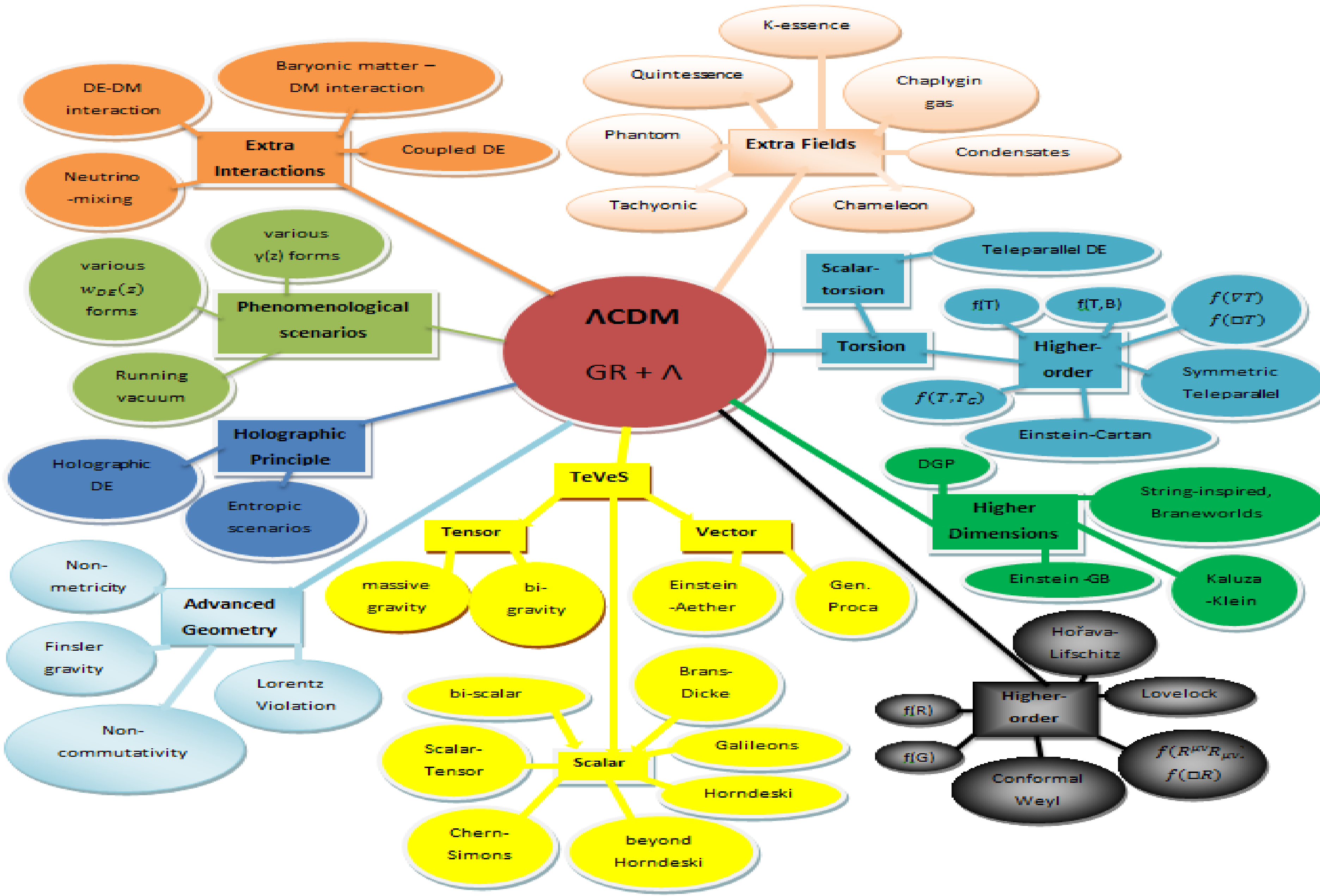
- Minimal matter coupling

General Relativity Assumptions and Considerations

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - 2\Lambda\right] + \int d^4x\, L_m\left(g_{\mu\nu}, \psi\right)$$

- Diffeomorphism invariance
- Spacetime dimensionality=4
- Geometry=Curvature (connection=Levi Civita)
- Linear in Ricci scalar
- Metric compatibility (zero non-metricity)
- Minimal matter coupling
- Equivalence principle
- Lorentz invariance
- Locality

Modified Gravity



Scalar-Tensor Theories

- Add a scalar field:

$$\boxed{L = \frac{1}{16\pi}\sqrt{-g}\left[f(\phi)R - s(\phi)\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda(\phi)\right] + L_m\left(h(\phi)g_{\mu\nu},\psi\right)}$$

Conformal Transf. to Jordan frame: $h(\phi)g_{\mu\nu} \rightarrow g_{\mu\nu}$

Scalar-Tensor Theories

- Add a scalar field:

$$\boxed{L = \frac{1}{16\pi}\sqrt{-g}\left[f(\phi)R - s(\phi)\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda(\phi)\right] + L_m\left(h(\phi)g_{\mu\nu}, \psi\right)}$$

Conformal Transf. to Jordan frame: $h(\phi)g_{\mu\nu} \to g_{\mu\nu}$

- Redefinition of ϕ :

$$L = \frac{1}{16\pi}\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi - 2V(\phi)\right] + L_m\left(g_{\mu\nu}, \psi\right)$$

- Brans-Dicke for $\omega \to const.,\ V \to 0$

- GR for $\omega \to \infty,\ \omega'/\omega^2 \to 0,\ V \to const.$

Brans-Dicke Theory

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\,, \tag{8.3}$$

$$\delta\left(\sqrt{-g}\,R\right) = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right)\delta g^{\mu\nu} \equiv \sqrt{-g}\,G_{\mu\nu}\,\delta g^{\mu\nu}\,, \tag{8.4}$$

yield the field equation

$$G_{\mu\nu} = \frac{8\pi}{\phi}\,T^{(m)}_{\mu\nu} + \frac{\omega}{\phi^2}\left(\nabla_\mu\phi\,\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\,\Box\phi\right) - \frac{V}{2\phi}\,g_{\mu\nu}\,, \tag{8.5}$$

where

$$T^{(m)}_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\,\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{-g}\,\mathcal{L}^{(m)}\right) \tag{8.6}$$

is the energy-momentum tensor of ordinary matter. By varying the action with respect to ϕ, one obtains

$$\frac{2\omega}{\phi}\,\Box\phi + R - \frac{\omega}{\phi^2}\,\nabla^\alpha\phi\nabla_\alpha\phi - \frac{dV}{d\phi} = 0\,. \tag{8.7}$$

Brans-Dicke Theory

Taking now the trace of Eq. (8.5),

$$R = \frac{-8\pi T^{(m)}}{\phi} + \frac{\omega}{\phi^2} \nabla^\alpha \phi \nabla_\alpha \phi + \frac{3\Box\phi}{\phi} + \frac{2V}{\phi}, \tag{8.8}$$

and using the resulting Eq. (8.8) to eliminate R from Eq. (8.7) leads to

$$\Box\phi = \frac{1}{2\omega+3}\left(8\pi T^{(m)} + \phi \frac{dV}{d\phi} - 2V\right). \tag{8.9}$$

According to this equation, the scalar ϕ is sourced by non-conformal matter (*i.e.*, by matter with trace $T^{(m)} \neq 0$), however the scalar does not couple directly to $\mathcal{L}^{(m)}$: the Brans-Dicke scalar ϕ reacts on ordinary matter only indirectly through the metric tensor $g_{\mu\nu}$, as dictated by Eq. (8.5). The term proportional to $\phi\, dV/d\phi - 2V$ on the right hand side of Eq. (8.9) vanishes if the potential has the form $V(\phi) = m^2\phi^2/2$ familiar from the Klein-Gordon equation and from particle physics. The action (8.1) and the field equation (8.5) suggest that the field ϕ be identified with the inverse of the effective gravitational coupling

$$G_{eff}(\phi) = \frac{1}{\phi}, \tag{8.10}$$

Scalar-Tensor Theories

- Field equations:

$$\phi G_{\mu\nu} + \left[\Diamond \phi + \frac{\omega}{2\phi} (\nabla \phi)^2 + V \right] g_{\mu\nu} - \nabla_\mu \nabla_\nu \phi - \frac{\omega}{\phi} \nabla_\mu \phi \nabla_\nu \phi = 8\pi \, T_{\mu\nu}$$

$$(2\omega + 3) \Box \phi + \omega' (\nabla \phi)^2 + 4V - 2\phi V' = 8\pi \, T$$

- For Brans-Dicke:
  - PPN parameters: $\beta_{PPN} = 1, \;\; \gamma_{PPN} = \dfrac{1+\omega}{2+\omega} \;\; \Rightarrow \;\; \omega \geq 40000$
  - Newton's constant: $G = \left(\dfrac{4 + 2\omega}{3 + 2\omega} \right) \dfrac{1}{\phi}$ with $\dfrac{\dot{G}}{G} \leq 1.7 \; 10^{-12} \; yr^{-1}$

Brans-Dicke Cosmology

- Friedmann-Robertson-Walker metric: $ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$
- Friedmann equations:

$$H^2 = \frac{8\pi}{3\phi}\rho_m - H\frac{\dot{\phi}}{\phi} + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2} + \frac{V}{3\phi}$$

$$2\dot{H} + 3H^2 = -\frac{1}{\phi}\left(8\pi p_m + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi} + 2H\dot{\phi} + \ddot{\phi}\right) + \frac{V}{\phi}$$

- Scalar-field equation:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{8\pi}{2\omega+3}\left(\rho_m - 3p_m\right) = 0 + \frac{2}{2\omega+3}\left(2V - \phi\frac{dV}{d\phi}\right)$$

- Matter equation: $\dot{\rho}_m + 3H\left(\rho_m + p_m\right) = 0$

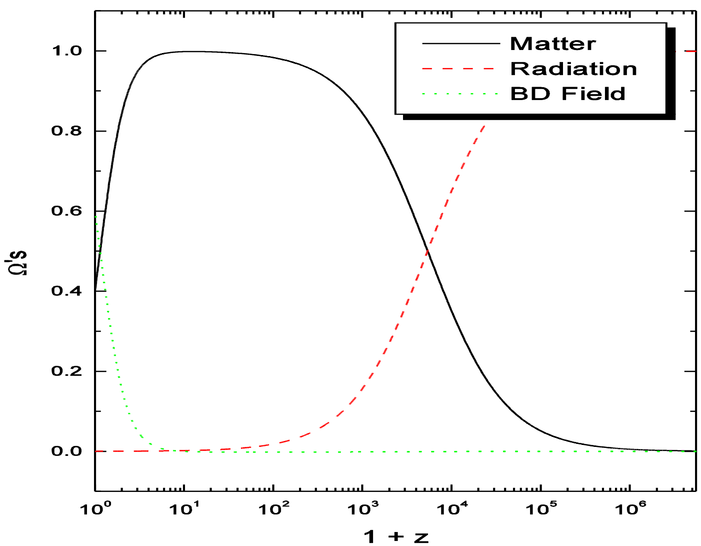
Dark Energy in Brans-Dicke Cosmology

- Effective Dark Energy sector:

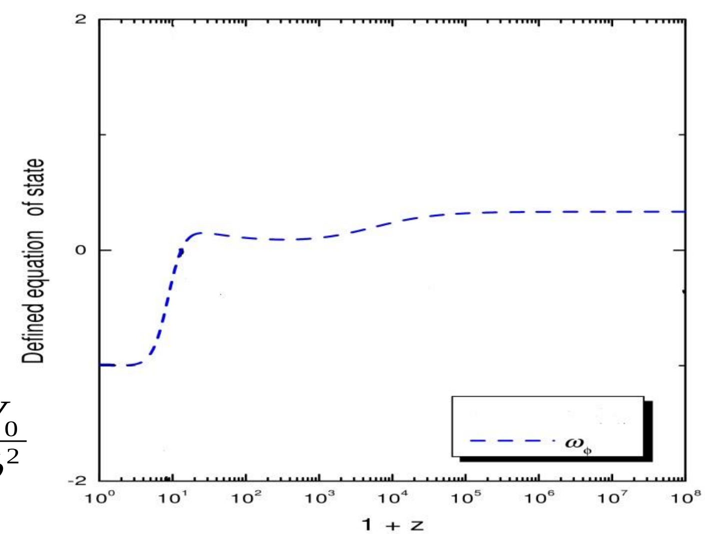
$$\rho_{DE} = \frac{3}{8\pi}\left(-H\dot{\phi} + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi}\right) + \frac{V}{8\pi}$$

$$p_{DE} = \frac{1}{8\pi}\left(\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi} + 2H\dot{\phi} + \ddot{\phi}\right) - \frac{V}{8\pi}$$

$$\Rightarrow w_{DE} = \frac{p_{DE}}{\rho_{DE}}$$



$$V(\phi) = \frac{V_0}{\phi^2}$$



Scalar-Tensor Theories

- Most general 4D scalar-tensor theories having second-order field equations:

$$L_H = \sum_{i=2}^{5} L_i$$

$$L_2[K] = K(\phi, X)$$

$$L_3[G_3] = -G_3(\phi, X)\Box\phi$$

$$X = -\partial^\mu\phi\partial_\mu\phi/2$$

$$L_4[G_4] = G_4(\phi, X)R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right]$$

$$L_5[G_5] = G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right]$$

[G. Horndeski, Int. J. Theor. Phys. 10]

Horndeski Theories

- Most general 4D scalar-tensor theories having second-order field equations:

$$\boxed{L_H = \sum_{i=2}^{5} L_i}$$

$$L_2[K] = K(\phi, X)$$

$$L_3[G_3] = -G_3(\phi, X)\Diamond\phi$$

$$X = -\partial^\mu\phi\partial_\mu\phi/2$$

$$L_4[G_4] = G_4(\phi, X)R + G_{4,X}\left[(\Diamond\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right]$$

$$L_5[G_5] = G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}\left[(\Diamond\phi)^3 - 3(\Diamond\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right]$$

[G. Horndeski, Int. J. Theor. Phys. 10]

- Coincides with Generalized Galileon theories

$$\phi \to \phi + c, \;\; \partial_\mu\phi \to \partial_\mu\phi + b_\mu$$

[Nicolis,Rattazzi,Trincherini, PRD 79]

Horndeski Cosmology (background)

- Field Equations: $L.H.S = R.H.S$
- In flat FRW:
- $$\boxed{\begin{aligned}&2XK_{,X} - K + 6X\dot{\phi}HG_{3,X} - 2XG_{3,\phi} - 6H^2G_4 + 24H^2X(G_{4,X} + XG_{4,XX}) - 12HX\dot{\phi}G_{4,\phi X} - 6H\dot{\phi}G_{4,\phi} \\ &+ 2H^3X\dot{\phi}(5G_{5,X} + 2XG_{5,XX}) - 6H^2X(3G_{5,\phi} + 2XG_{5,\phi X}) = -\rho_m\end{aligned}}$$
- $$\boxed{\begin{aligned}&K - 2X(G_{3,\phi} + \ddot{\phi}G_{3,X}) + 2(3H^2 + 2\dot{H})G_4 - 12H^2XG_{4,X} - 4H\dot{X}G_{4,X} - 8\dot{H}XG_{4,X} - 8HX\dot{X}G_{4,XX} \\ &+ 2(\ddot{\phi} + 2H\dot{\phi})G_{4,\phi} + 4XG_{4,\phi\phi} + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4,\phi X} - 2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5,X} - 4H^2X^2\ddot{\phi}G_{5,XX} \\ &+ 4HX(\dot{X} - HX)G_{5,\phi X} + 2\left[2(\dot{H}X + H\dot{X}) + 3H^2X\right]G_{5,\phi} + 4HX\dot{\phi}G_{5,\phi\phi} = -p_m\end{aligned}}$$
- $$\boxed{\frac{1}{a^3}\frac{d}{dt}(a^3J) = P_\phi}$$

with
$$J = \dot{\phi}K_{,X} + 6HXG_{3,X} - 2\dot{\phi}G_{3,\phi} + 6H^2\dot{\phi}(G_{4,X} + 2XG_{4,XX}) - 12HXG_{4,\phi X} + 2H^3X(3G_{5,X} + 2XG_{5,XX}) - 6H^2\dot{\phi}(G_{5,\phi} + XG_{5,\phi X})$$
$$P_\phi = K_{,\phi} - 2X(G_{3,\phi\phi} + \ddot{\phi}G_{3,\phi X}) + 6(2H^2 + \dot{H})G_{4,\phi} + 6H(\dot{X} + 2HX)G_{4,\phi X} - 6H^2XG_{5,\phi\phi} + 2H^3X\dot{\phi}G_{5,\phi X}$$

[De Felice,Tsujikawa JCAP 1202]

Horndeski Cosmology (perturbations)

- Scalar perturbations: $ds^2 = -(1+2\psi)dt^2 + a^2(1-2\phi)\delta_{ij}dx^i dx^j$ $\qquad \Rightarrow L.H.S = R.H.S$
- No-ghost condition: $\boxed{Q_S \equiv \frac{w_1\left(4w_1w_3 + 9w_2^2\right)}{3w_2^2} > 0}$
- No Laplacian instabilities condition: $\boxed{c_S^2 \equiv \frac{3\left(2w_1^2w_2H - 4w_2^2w_4 + 4w_1w_2\dot{w}_1 - 2w_1^2\dot{w}_2\right) - 6w_1^2\left(\rho_m + p_m\right)}{w_1\left(4w_1w_3 + 9w_2^2\right)} > 0}$

with $\quad \mathrm{w}_1 \equiv 2\left(G_4 - 2XG_{4,X}\right) - 2X\left(G_{5,X}\dot{\phi}H - G_{5,\phi}\right)$

$$\mathrm{w}_2 \equiv -2G_{3,X}X\dot{\phi} + 4G_4H - 16X^2G_{4,XX}H + 4\left(\dot{\phi}G_{4,\phi X} - 4HG_{4,X}\right)X + 2G_{4,\phi}\dot{\phi}$$
$$+ 8X^2G_{5,\phi X}H + 2HX\left(6G_{5,\phi} - 5HG_{5,X}\dot{\phi}\right) - 4G_{5,XX}\dot{\phi}X^2H^2$$

$$\mathrm{w}_3 \equiv 3X\left(K_{,X} + 2XK_{,XX}\right) + 6X\left(3X\dot{\phi}HG_{3,XX} - G_{3,\phi X}X - G_{3,\phi} + 6\dot{\phi}HG_{3,X}\right)$$
$$+ 18H\left(4HX^3G_{4,XXX} - HG_4 - 5X\dot{\phi}G_{4,\phi X} - G_{4,\phi}\dot{\phi} + 7HG_{4,X}X + 16HX^2G_{4,XX} - 2X^2\dot{\phi}G_{4,X\phi X}\right)$$
$$+ 6H^2X\left(2H\dot{\phi}G_{5,XXX}X^2 - 6X^2G_{5,\phi XX} + 13XH\dot{\phi}G_{5,XX} - 27G_{5,\phi X}X + 15H\dot{\phi}G_{5,X} - 18G_{5,\phi}\right)$$

$$\mathrm{w}_4 \equiv 2G_4 - 2XG_{5,\phi} - 2XG_{5,X}\ddot{\phi}$$

Beyond Horndeski Theories

- Beyond Horndeski, free from Ostrogradski instabilities but still propagating 2+1 dof's:

$$\boxed{L_{BH} = \sum_{i=2}^{5} L_i}$$

$$X = -\partial^\mu \phi \partial_\mu \phi / 2$$

$$A_i = A_i(\phi, X)$$

$$B_i = B_i(\phi, X)$$

$$L_2 = L_2^H[A_2]$$

$$L_3 = L_3^H[C_3 + 2XC_{3,X}] + L_2^H[XC_{3,\phi}]$$

$$L_4 = L_4^H[B_4] + L_3^H[C_4 + 2XC_{4,X}] + L_2^H[XC_{4,\phi}] - \frac{B_4 + A_4 - 2XB_{4,X}}{X^2} L^{gal1}$$

$$L_5 = L_5^H[G_4] + L_4^H[C_5] + L_3^H[D_5 + 2XD_{5,X}] + L_2^H[XD_{5,\phi}] + \frac{XB_{5,X} + 3A_5}{3(-X)^{5/2}} L^{gal2}$$

with

$$L^{gal1} = X\left[(\Diamond\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right] - 2\left[(\nabla^\mu\phi\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)(\Diamond\phi) - (\nabla^\mu\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla_\lambda\phi)(\nabla^\lambda\nabla^\nu\phi)\right]$$

$$L^{gal2} = X\left[(\Diamond\phi)^3 - 3(\Diamond\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla_\mu\nabla_\nu\phi)(\nabla^\nu\nabla^\rho\phi)(\nabla^\mu\nabla_\rho\phi)\right]$$
$$-3\begin{bmatrix}(\Diamond\phi)^2(\nabla_\mu\phi)(\nabla^\mu\nabla^\nu\phi)(\nabla_\nu\phi) - 2(\Diamond\phi)(\nabla_\mu\phi)(\nabla^\mu\nabla^\nu\phi)(\nabla_\nu\nabla_\rho\phi)(\nabla^\rho\phi) \\ -(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)(\nabla_\rho\phi)(\nabla^\rho\nabla^\lambda\phi)(\nabla_\lambda\phi) + 2(\nabla_\mu\phi)(\nabla^\mu\nabla^\nu\phi)(\nabla_\nu\nabla_\rho\phi)(\nabla^\rho\nabla^\lambda\phi)(\nabla_\lambda\phi)\end{bmatrix}$$

$$C_3 = \frac{1}{2}\int A_3(-X)^{-3/2} dX \qquad C_5 = -\frac{1}{4} X \int B_{5,\phi}(-X)^{-3/2} dX$$

$$C_4 = -\int B_{4,\phi}(-X)^{-1/2} dX \qquad D_5 = -\int C_{5,\phi}(-X)^{-1/2} dX \qquad G_5 = -\int B_{5,X}(-X)^{-1/2} dX$$

- Primary constraint prevents the propagation of extra degrees of freedom

[Gleyzes,Langlois,Piazza,Vernizzi, PRL 114], [Crisostomi,Hull,Koyama,Tasinato, JCAP 1603]

Inflation: scalar field

$$\boxed{\mathrm{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \mathrm{V}(\varphi)}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,,$$

$$H^2 = \frac{8\pi}{3m_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi(\phi) = 0$$

Inflation: scalar field

$$\boxed{\mathrm{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \mathrm{V}(\varphi)}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,,$$

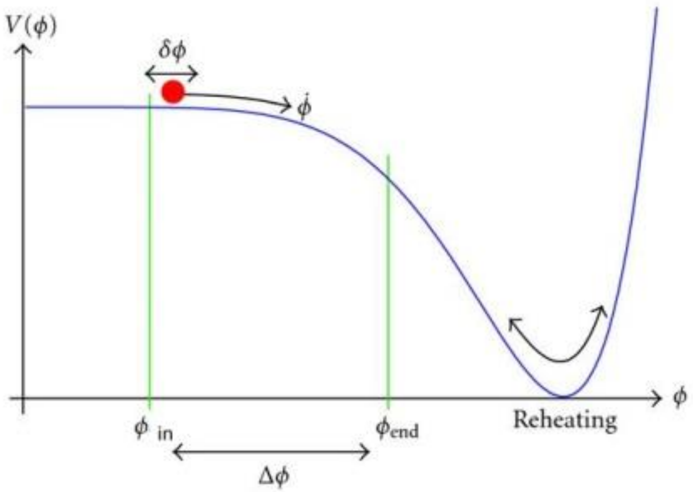
$$H^2 = \frac{8\pi}{3m_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi(\phi) = 0$$

- Slow-roll conditions: $\dot{\phi}^2/2 \ll V(\phi)$ and $|\ddot{\phi}| \ll 3H|\dot{\phi}|$

$$H^2 \simeq \frac{8\pi V(\phi)}{3m_{\rm pl}^2}\,,$$

$$3H\dot{\phi} \simeq -V_\phi(\phi)$$



$$N \equiv \ln\frac{a_f}{a} = \int_t^{t_f} H\mathrm{d}t \simeq \frac{8\pi}{m_{\rm pl}^2}\int_{\phi_f}^{\phi}\frac{V}{V_\phi}\mathrm{d}\phi$$

Primordial Spectra

The results for the power spectra of the scalar and tensor fluctuations created by inflation are

$$\Delta_{\mathrm{s}}^2(k) \equiv \Delta_{\mathcal{R}}^2(k) = \left. \frac{1}{8\pi^2} \frac{H^2}{M_{\mathrm{pl}}^2} \frac{1}{\varepsilon} \right|_{k=aH} , \tag{222}$$

$$\Delta_{\mathrm{t}}^2(k) \equiv 2\Delta_h^2(k) = \left. \frac{2}{\pi^2} \frac{H^2}{M_{\mathrm{pl}}^2} \right|_{k=aH} , \tag{223}$$

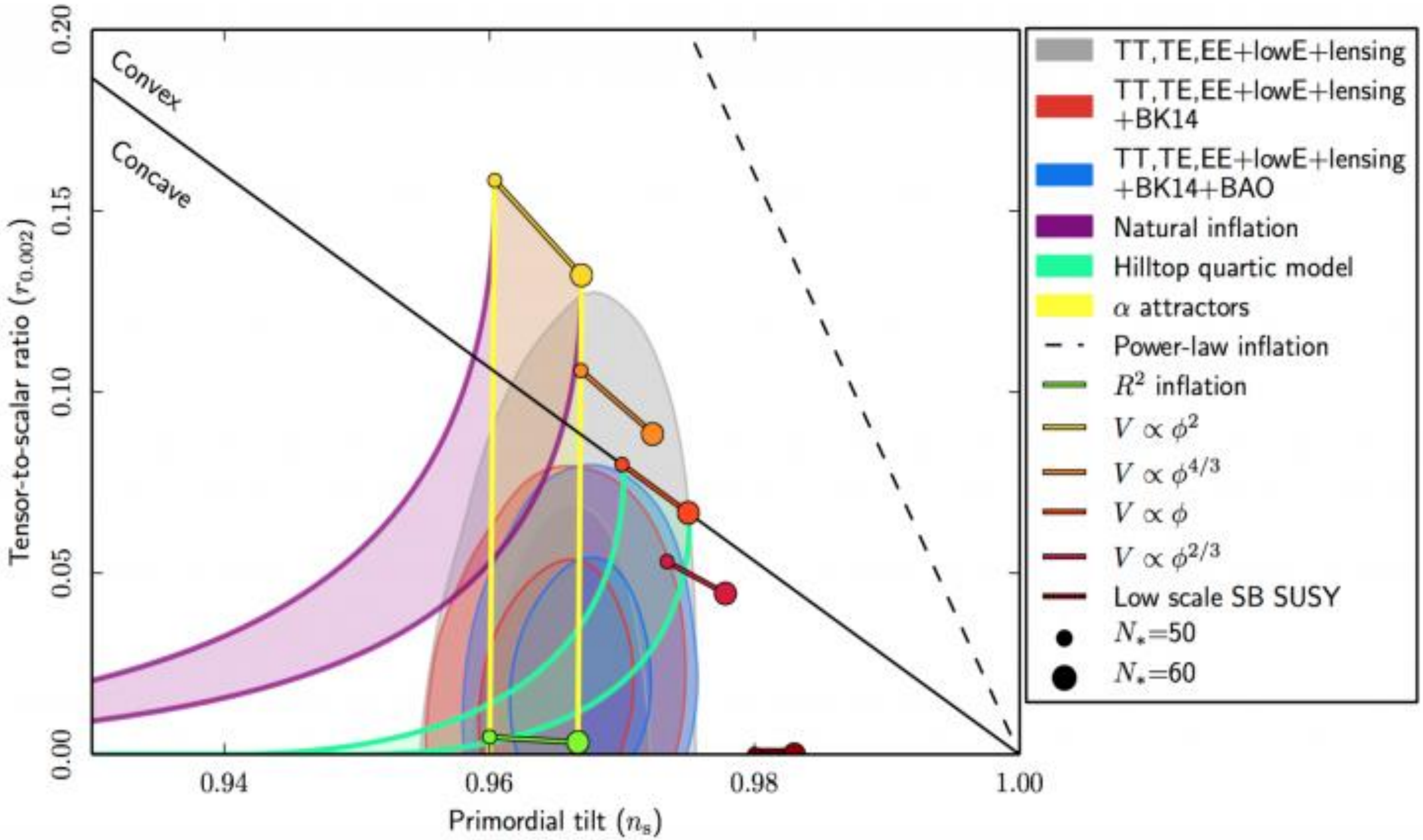
where

$$\varepsilon = -\frac{d \ln H}{dN} . \tag{224}$$

The horizon crossing condition $k = aH$ makes (222) and (223) functions of the comoving wavenumber k. The tensor-to-scalar ratio is

$$r \equiv \frac{\Delta_{\mathrm{t}}^2}{\Delta_{\mathrm{s}}^2} = 16\, \varepsilon_\star . \tag{225}$$

Simple Inflation models: problem



Inflation in Nonminimal Derivative Coupling

- $$\boxed{S=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R-\frac{1}{2}(g_{\mu\nu}-\zeta G_{\mu\nu})\partial^\mu\phi\partial^\nu\phi-V(\phi)\right]+S_m+S_r}$$

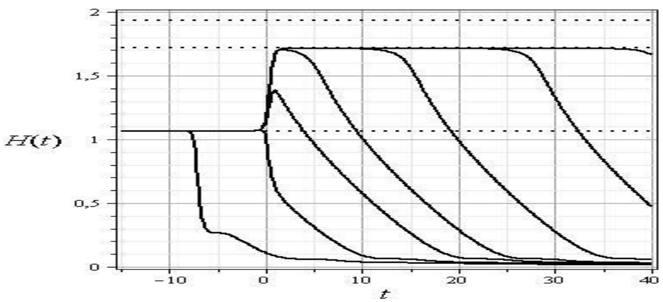
- In flat FRW:

$$H^2=\frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2}\left(1+9\zeta H^2\right)+V(\phi)+\rho_m+\rho_r\right]$$

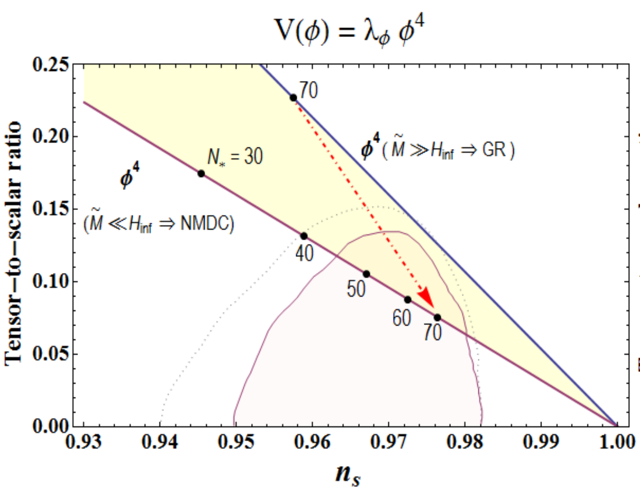
$$2\dot{H}+3H^2=-8\pi G\left[\frac{\dot{\phi}^2}{2}\left[1-\zeta\left(2\dot{H}+3H^2+\frac{4H\ddot{\phi}}{\dot{\phi}}\right)\right]-V(\phi)+p_m+p_r\right]$$

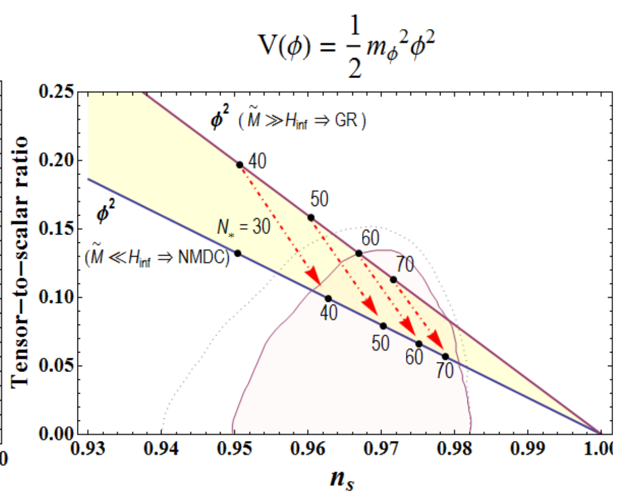
Inflation with Nonminimal Derivative Coupling

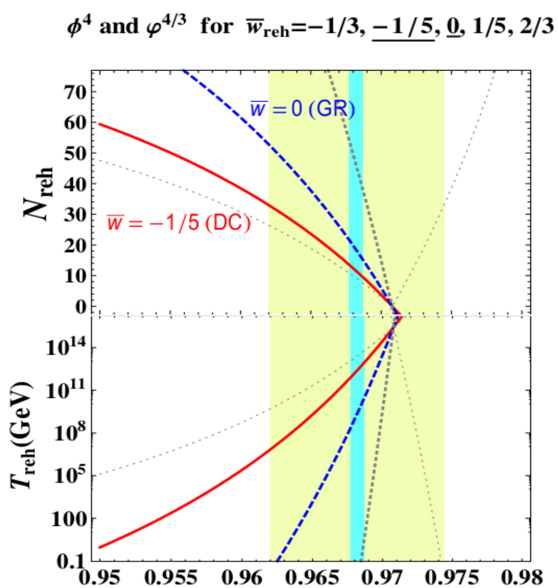
- New Higgs Inflation: $r \approx 0.05$



$$V(\phi) = V_0 \phi^2$$







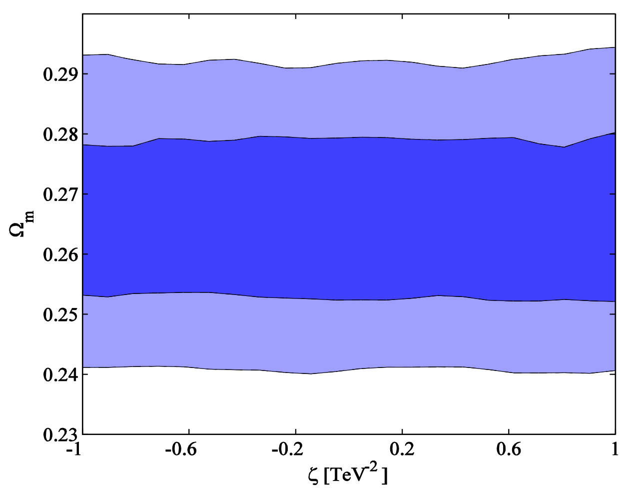
Dark-Energy in Nonminimal Derivative Coupling

- $$S=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R-\frac{1}{2}(g_{\mu\nu}-\zeta G_{\mu\nu})\partial^\mu\phi\partial^\nu\phi-V(\phi)\right]+S_m+S_r$$

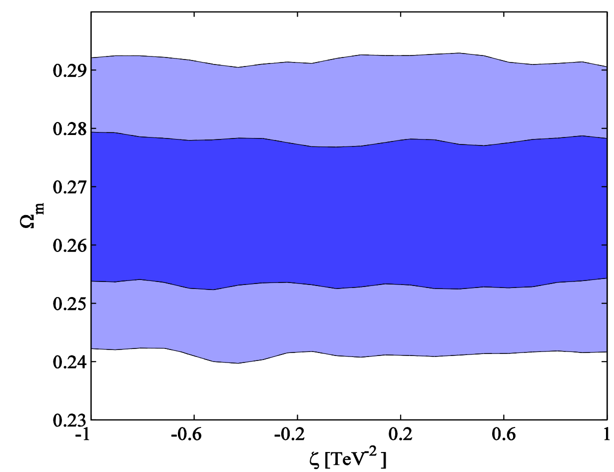
- In flat FRW:

$$H^2=\frac{8\pi G}{3}\left[\frac{\dot\phi^2}{2}\left(1+9\zeta H^2\right)+V(\phi)+\rho_m+\rho_r\right]$$

$$2\dot H+3H^2=-8\pi G\left[\frac{\dot\phi^2}{2}\left[1-\zeta\left(2\dot H+3H^2+\frac{4H\ddot\phi}{\dot\phi}\right)\right]-V(\phi)+p_m+p_r\right]$$



$$V(\phi)=V_0e^{\lambda\phi}$$

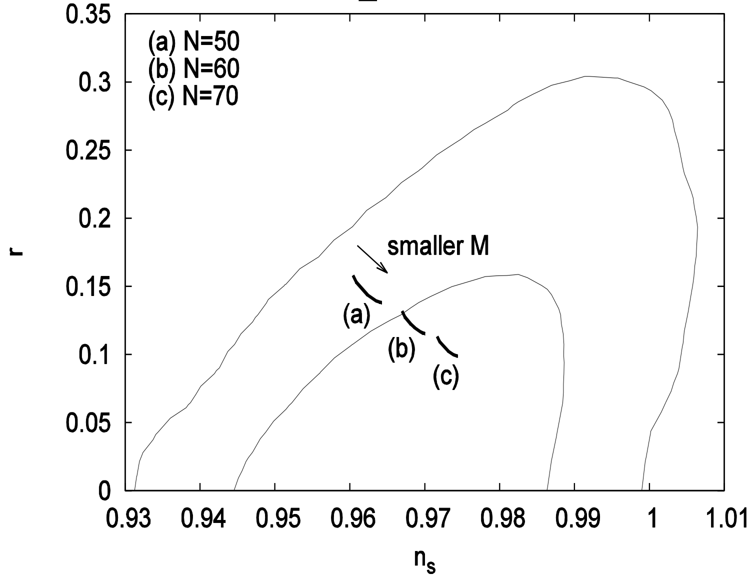


$$V(\phi)=V_0\phi^n$$

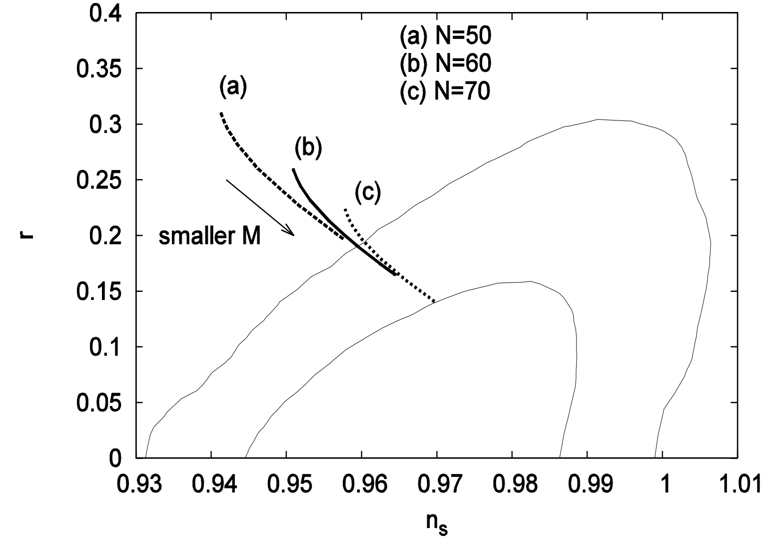
Inflation in Horndeski Theories

- $K(\phi, X) = X - V(\phi),\ G_3(\phi, X) = \frac{c_3}{M^3} X,\ G_4 = G_5 = 0$

$V(\phi) = \frac{1}{2} m^2 \phi^2$



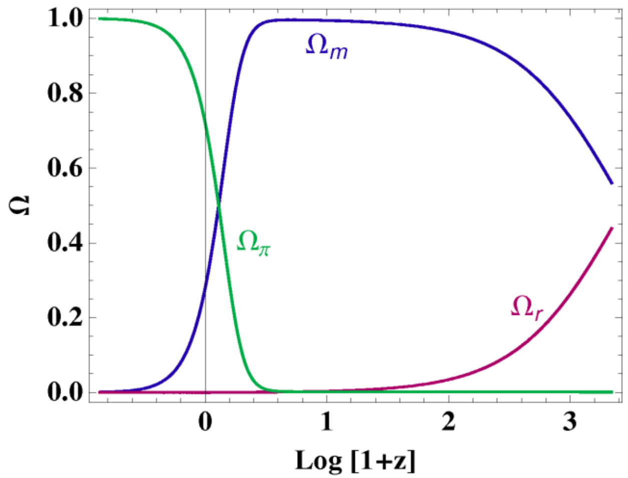
$V(\phi) = \frac{1}{4} \lambda \phi^4$



- G-Inflation (Shift-symmetric): $K(\phi, X) = X + \frac{X^2}{2M^3\mu},\ G_3(\phi, X) = \frac{1}{M^3} X,\ G_4 = G_5 = 0$

Dark Energy in Horndeski Theories

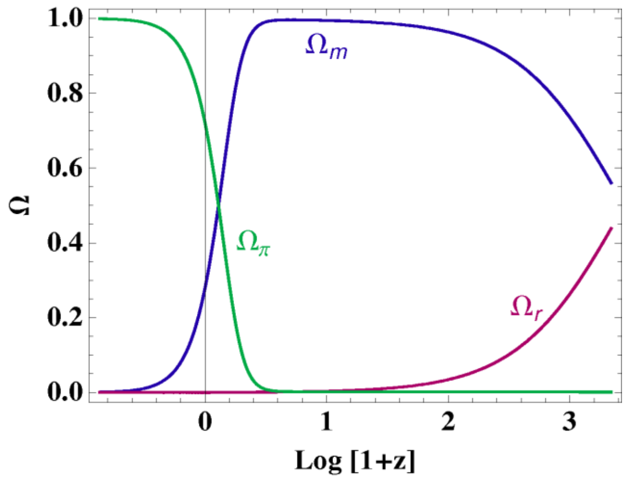
$$K(\phi, X) = c_2 X\,,\ \ G_3(\phi, X) = c_3,\ \ G_4 = 1,\ \ G_5 = c_5$$



Dark Energy in Horndeski Theories

- $K(\phi, X) = c_2 X,\ G_3(\phi, X) = c_3,\ G_4 = 1,\ G_5 = c_5$

- Background evolution: Universe thermal history

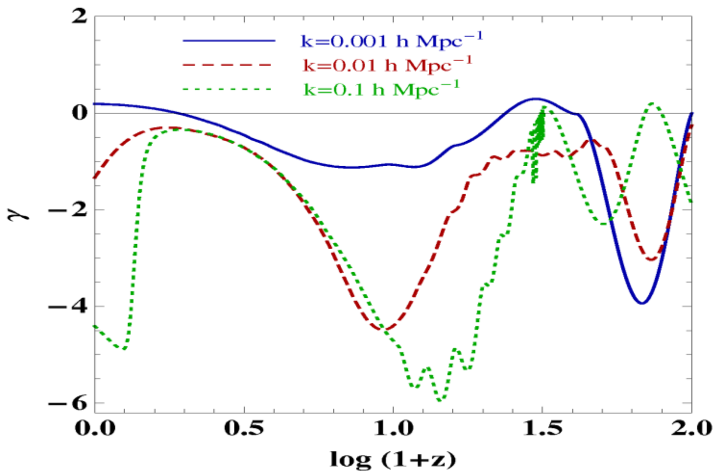


- Perturbations: $\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G_{eff}\rho_m\delta_m$

  with $G_{eff} = G_{eff}(\phi, K, G_3, G_4, G_5)$

- Clustering growth rate: $\boxed{\frac{d\ln\delta_m}{d\ln a} = \Omega_m^\gamma(a)}$

  γ(z): Growth index.



R^2 gravity

1. *The case of* $f(R) = R + \alpha R^2$

Quadratic corrections in the Ricci scalar motivated by attempts to renormalize GR, constitute a straightforward extension of GR and have been particularly relevant in cosmology since they allow a self-consistent inflationary model to be constructed [530]. We will use this model as an example before discussing general metric $f(R)$-gravity.

Let us begin by deriving the field equations for the Lagrangian density

$$\mathcal{L} = R + \alpha R^2 + 2\kappa \mathcal{L}^{(m)} \tag{8.11}$$

from the variational principle $\delta \int d^4x \sqrt{-g}\, \mathcal{L} = 0$. We consider vacuum first. The variation gives

$$\int d^4x \sqrt{-g}\, G_{\alpha\beta} \delta g^{\alpha\beta} + \alpha\, \delta \int d^4x \sqrt{-g}\, R^2 = 0\,, \tag{8.12}$$

in which the variation of $R\sqrt{-g}$ produces the Einstein tensor. We now compute the second term on the right hand side of Eq. (8.12). We have

$$\delta \int d^4x \sqrt{-g}\, R^2 = -\frac{1}{2} \int d^4x \sqrt{-g}\; g_{\alpha\beta} \delta g^{\alpha\beta} R^2 + 2 \int d^4x \sqrt{-g}\, R \delta R \tag{8.13}$$

and

$$\int d^4x \sqrt{-g}\, R \delta R = \int d^4x \sqrt{-g}\, R \left(\delta g^{\alpha\beta} R_{\alpha\beta} + g^{\alpha\beta} \delta R_{\alpha\beta} \right) . \tag{8.14}$$

R^2 gravity

By using the fact that

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_\alpha \nabla_\beta h^{\alpha\beta} - \Box h\,, \tag{8.15}$$

where

$$h^{\alpha\beta} \equiv -\delta g^{\alpha\beta}, \qquad h \equiv -g_{\alpha\beta}\delta g^{\alpha\beta}\,, \tag{8.16}$$

one has

$$\int d^4x\sqrt{-g}\,R\,g^{\alpha\beta}\delta R_{\alpha\beta} = \int d^4x\sqrt{-g}\,R\left(\nabla_\alpha\nabla_\beta h^{\alpha\beta} - \Box h\right)\,. \tag{8.17}$$

Integrating by parts twice, the operators $\nabla_\alpha\nabla_\beta$ and $\Box$ acting on $h^{\alpha\beta}$ and h, respectively, transfer their action onto R and

$$\int d^4x\sqrt{-g}\,R\,g^{\alpha\beta}\delta R_{\alpha\beta} = \int d^4x\sqrt{-g}\,\left(h^{\alpha\beta}\nabla_\alpha\nabla_\beta R - h\Box R\right)\,. \tag{8.18}$$

R^2 gravity

Using Eq. (8.16), Eq. (8.18) becomes

$$\int d^4x\sqrt{-g}\,R\,g^{\alpha\beta}\delta R_{\alpha\beta} = \int d^4x\sqrt{-g}\left(-\delta g^{\alpha\beta}\nabla_\alpha\nabla_\beta R + g_{\alpha\beta}\Box R\delta g^{\alpha\beta}\right) . \tag{8.19}$$

Upon substitution of Eq. (8.19) into Eq. (8.14), one obtains

$$\int d^4x\sqrt{-g}\,R\delta R = \int d^4x\,\sqrt{-g}\left(R\delta g^{\alpha\beta}R_{\alpha\beta} - \delta g^{\alpha\beta}\nabla_\alpha\nabla_\beta R + g_{\alpha\beta}\Box R\delta g^{\alpha\beta}\right) \tag{8.20}$$

and Eq. (8.13) takes the form

$$\begin{aligned}\delta\int d^4x\sqrt{-g}\,R^2 &= -\frac{1}{2}\int d^4x\sqrt{-g}\,g_{\alpha\beta}\delta g^{\alpha\beta}R^2 + 2\int d^4x\sqrt{-g}\,\left(R\delta g^{\alpha\beta}R_{\alpha\beta} - \delta g^{\alpha\beta}\nabla_\alpha\nabla_\beta R + g_{\alpha\beta}\Box R\delta g^{\alpha\beta}\right)\\ &= \int d^4x\sqrt{-g}\left(2RR_{\alpha\beta} - \tfrac{1}{2}\,g_{\alpha\beta}R^2\right)\delta g^{\alpha\beta} + 2\int d^4x\sqrt{-g}\,\left(g_{\alpha\beta}\Box R - \nabla_\alpha\nabla_\beta R\right)\delta g^{\alpha\beta} .\end{aligned} \tag{8.21}$$

R^2 gravity

Substituting this equation into Eq. (8.12) and including the matter part of the Lagrangian $\mathcal{L}^{(m)}$ which produces the energy-momentum tensor $T_{\mu\nu}^{(m)}$, the field equations

$$G_{\alpha\beta} + \alpha\left[2R\left(R_{\alpha\beta} - \frac{1}{4}g_{\alpha\beta}R\right) + 2\left(g_{\alpha\beta}\Box R - \nabla_\alpha\nabla_\beta R\right)\right] = \kappa T_{\alpha\beta}^{(m)} \tag{8.22}$$

are obtained; they are fourth-order equations for the metric components. The trace of Eq. (8.22) is

$$\Box R - \frac{1}{6\alpha}\left(R + \kappa T^{(m)}\right) = 0\,, \tag{8.23}$$

R^2 gravity

which shows that α must be positive. One can also define an angular frequency ω (equivalent to a mass m) so that

$$\frac{1}{6\alpha} = \omega^2 = m^2 \,. \tag{8.24}$$

Following this definition, Eq. (8.23) becomes

$$\Box R - m^2 \left(R + \kappa T^{(m)} \right) = 0 \,. \tag{8.25}$$

Eq. (8.25) can be seen as an effective Klein-Gordon equation for the effective scalar field degree of freedom R (sometimes called *scalaron*).

f(R) gravity

2. $f(R)$-gravity: the general case

Let us discuss now a generic analytical[15] function $f(R)$ in the metric formalism, beginning with the vacuum case, as described by the Lagrangian density $\sqrt{-g}\,\mathcal{L} = \sqrt{-g}\,f(R)$ obeying the variational principle $\delta\int d^4x\sqrt{-g}\,f(R) = 0$. We have

$$\begin{aligned}\delta\int d^4x\sqrt{-g}\,f(R) &= \int d^4x\left[\delta\left(\sqrt{-g}\,f(R)\right) + \sqrt{-g}\,\delta\left(f(R)\right)\right]\\ &= \int d^4x\sqrt{-g}\left[f'(R)R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}f(R)\right]\delta g^{\mu\nu} + \int d^4x\sqrt{-g}\,f'(R)g^{\mu\nu}\delta R_{\mu\nu}\,,\end{aligned} \tag{8.26}$$

where the prime denotes differentiation with respect to R. We now compute these integrals in the local inertial frame. By using

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\partial_\sigma\left(\delta G^{\sigma}_{\mu\nu}\right) - g^{\mu\sigma}\partial_\sigma\left(\delta G^{\nu}_{\mu\nu}\right) \equiv \partial_\sigma W^\sigma \tag{8.27}$$

---

[15] This assumption is not, strictly speaking, necessary and is sometimes relaxed in the literature.

f(R) gravity

where

$$W^{\sigma} \equiv g^{\mu\nu}\delta G^{\sigma}_{\mu\nu} - g^{\mu\sigma}\delta G^{\nu}_{\mu\nu}\,, \tag{8.28}$$

the second integral in Eq. (8.26) can be written as

$$\int d^4x\sqrt{-g}\,f'(R)g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}\,f'(R)\partial_{\sigma}W^{\sigma}\,. \tag{8.29}$$

Integration by parts yields

$$\int d^4x\sqrt{-g}\,f'(R)g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\frac{\partial}{\partial x^{\sigma}}\left[\sqrt{-g}\,f'(R)W^{\sigma}\right] - \int d^4x\partial_{\sigma}\left[\sqrt{-g}f'(R)\right]W^{\sigma}\,. \tag{8.30}$$

The first integrand is a total divergence and can be discarded by assuming that the fields vanish at infinity, obtaining

$$\int d^4x\sqrt{-g}\,f'(R)g^{\mu\nu}\delta R_{\mu\nu} = -\int d^4x\partial_{\sigma}\left[\sqrt{-g}\,f'(R)\right]W^{\sigma}\,. \tag{8.31}$$

Let us calculate now the term W^{σ} appearing in Eq. (8.31). We have

$$\delta G^{\sigma}_{\mu\nu} = \delta\left[\frac{1}{2}g^{\sigma\alpha}\left(\partial_{\mu}g_{\alpha\nu} + \partial_{\nu}g_{\mu\alpha} - \partial_{\alpha}g_{\mu\nu}\right)\right] = \frac{1}{2}g^{\sigma\alpha}\left[\partial_{\mu}\left(\delta g_{\alpha\nu}\right) + \partial_{\nu}\left(\delta g_{\mu\alpha}\right) - \partial_{\alpha}\left(\delta g_{\mu\nu}\right)\right]\,, \tag{8.32}$$

since in the locally inertial frame considered here it is

$$\partial_{\alpha}g_{\mu\nu} = \nabla_{\alpha}g_{\mu\nu} = 0\,. \tag{8.33}$$

f(R) gravity

Similarly, it is

$$\delta G^{\nu}_{\mu\nu} = \frac{1}{2}\, g^{\nu\alpha}\partial_\mu \left(\delta g_{\nu\alpha}\right) \ . \tag{8.34}$$

By combining Eqs. (8.33) and (8.34), one obtains

$$g^{\mu\nu}\delta G^{\sigma}_{\mu\nu} = \frac{1}{2}\, g^{\mu\nu}\left[-\partial_\mu \left(g_{\alpha\nu}\delta g^{\alpha\sigma}\right) - \partial_\nu \left(g_{\mu\alpha}\delta g^{\sigma\alpha}\right) - g^{\sigma\alpha}\partial_\alpha \left(\delta g_{\mu\nu}\right)\right] = \frac{1}{2}\,\partial^\sigma \left(g_{\mu\nu}\delta g^{\mu\nu}\right) - \partial^\mu \left(g_{\alpha\mu}\delta g^{\nu\alpha}\right) \ , \tag{8.35}$$

$$g^{\mu\sigma}\delta G^{\nu}_{\mu\nu} = -\frac{1}{2}\,\partial^\sigma \left(g_{\nu\alpha}\delta g^{\nu\alpha}\right) \ , \tag{8.36}$$

from which it follows immediately that

$$W^\sigma = \partial^\sigma \left(g_{\mu\nu}\delta g^{\mu\nu}\right) - \partial^\mu \left(g_{\mu\nu}\delta g^{\sigma\nu}\right) \ . \tag{8.37}$$

Using this equation one can write

$$\int d^4x\sqrt{-g}\, f'(R) g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x \partial_\sigma \left[\sqrt{-g}\, f'(R)\right]\left[\partial^\mu \left(g_{\mu\nu}\delta g^{\sigma\nu}\right) - \partial^\sigma \left(g_{\mu\nu}\delta g^{\mu\nu}\right)\right] \ . \tag{8.38}$$

Integrating by parts and discarding total divergences, one obtains

$$\int d^4x\,\sqrt{-g}\, f'(R) g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\, g_{\mu\nu}\partial^\sigma\partial_\sigma \left[\sqrt{-g} f'(R)\right]\delta g^{\mu\nu} - \int d^4x\, g_{\mu\nu}\partial^\mu\partial_\sigma \left[\sqrt{-g}\, f'(R)\right]\delta g^{\sigma\nu} \ . \tag{8.39}$$

f(R) gravity

The variation of the action is then

$$\delta \int d^4x \sqrt{-g}\, f(R) = \int d^4x \sqrt{-g} \left[f'(R) R_{\mu\nu} - \tfrac{1}{2} f(R) g_{\mu\nu} \right] \delta g^{\mu\nu}$$
$$+ \int d^4x \left[g_{\mu\nu} \partial^\sigma \partial_\sigma \left(\sqrt{-g}\, f'(R) \right) - g_{\sigma\nu} \partial^\mu \partial_\sigma \left(\sqrt{-g} f'(R) \right) \right] \delta g^{\mu\nu} . \tag{8.40}$$

The vanishing of the variation implies the fourth order vacuum field equations

$$f'(R) R_{\mu\nu} - \frac{f(R)}{2} g_{\mu\nu} = \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R) . \tag{8.41}$$

f(R) gravity

These equations can be re-arranged in the Einstein-like form

$$f'(R)R_{\mu\nu} - \frac{f'(R)}{2}\,g_{\mu\nu}R + \frac{f'(R)}{2}\,g_{\mu\nu}R - \frac{f(R)}{2}\,g_{\mu\nu} = \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\,, \tag{8.42}$$

and then

$$G_{\mu\nu} = \frac{1}{f'(R)}\left\{\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) + g_{\mu\nu}\frac{[f(R) - f'(R)R]}{2}\right\} \tag{8.43}$$

The right hand side of Eq. (8.43) is then regarded as an effective stress-energy tensor, which we call *curvature fluid* energy-momentum tensor $T_{\mu\nu}^{(curv)}$ sourcing the effective Einstein equations. Although this interpretation is questionable in principle because the field equations describe a theory different from GR, and one is forcing upon them the interpretation as effective Einstein equations, this approach is quite useful in practice.

f(R) gravity – Palatini approach

$$S_{pal} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\, f(\mathcal{R}) + S_M(g_{\mu\nu},\psi). \qquad (13)$$

$$\delta\mathcal{R}_{\mu\nu} \;=\; \bar{\nabla}_\lambda \delta\Gamma^\lambda{}_{\mu\nu} - \bar{\nabla}_\nu \delta\Gamma^\lambda{}_{\mu\lambda}. \qquad (14)$$

yields

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa T_{\mu\nu}, \qquad (15)$$

$$-\bar{\nabla}_\lambda\left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g} f'(\mathcal{R}) g^{\sigma(\mu}\right)\delta^{\nu)}_\lambda = 0, \qquad (16)$$

where $T_{\mu\nu}$ is defined in the usual way as in eq. (7), $\bar{\nabla}_\mu$ denotes the covariant derivative defined with the independent connection $\Gamma^\lambda{}_{\mu\nu}$, and $(\mu\nu)$ and $[\mu\nu]$ denote symmetrization or anti-symmetrization over the indices μ and ν, respectively. Taking the trace of eq. (16), it can be

f(R) gravity – Palatini approach

easily shown that

$$\bar{\nabla}_\sigma \left(\sqrt{-g} f'(\mathcal{R}) g^{\sigma\mu} \right) = 0, \tag{17}$$

which implies that we can bring the field equations into the more economical form

$$\begin{aligned} f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} &= \kappa G T_{\mu\nu}, & (18) \\ \bar{\nabla}_\lambda \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu} \right) &= 0, & (19) \end{aligned}$$

f(R) gravity – Palatini approach

Finally, let us present some useful manipulations of the field equations. Taking the trace of eq. (18) yields

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T. \tag{20}$$

As in the metric case, this equation will prove very useful later on. For a given f, it is an algebraic equation in $\mathcal{R}$. For all cases in which $T = 0$, including vacuum and electrovacuum, $\mathcal{R}$ will therefore be a constant and a root of the equation

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 0. \tag{21}$$

f(R) gravity – Palatini approach

$$h_{\mu\nu} \equiv f'(\mathcal{R}) g_{\mu\nu}. \tag{22}$$

$$\begin{aligned} \mathcal{R}_{\mu\nu} = R_{\mu\nu} \; &+ \; \frac{3}{2}\frac{1}{(f'(\mathcal{R}))^2}\left(\nabla_\mu f'(\mathcal{R})\right)\left(\nabla_\nu f'(\mathcal{R})\right) - \\ &- \; \frac{1}{f'(\mathcal{R})}\left(\nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu\nu} \Box\right) f'(\mathcal{R}). \end{aligned} \tag{26}$$

Contraction with $g^{\mu\nu}$ yields

$$\begin{aligned} \mathcal{R} = R \; &+ \; \frac{3}{2(f'(\mathcal{R}))^2}\left(\nabla_\mu f'(\mathcal{R})\right)\left(\nabla^\mu f'(\mathcal{R})\right) \\ &+ \; \frac{3}{f'(\mathcal{R})} \Box f'(\mathcal{R}). \end{aligned} \tag{27}$$

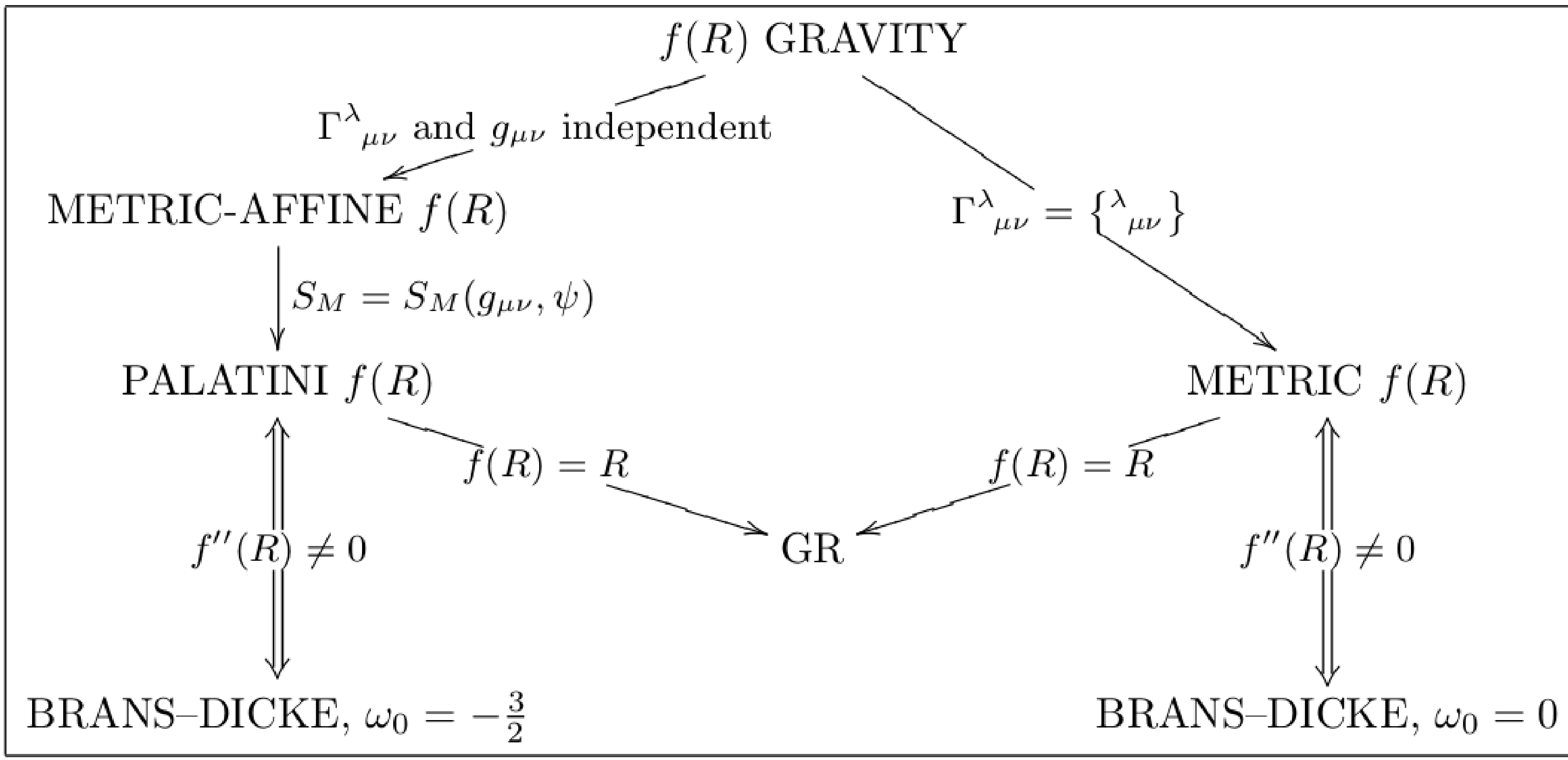
Note the difference between $\mathcal{R}$ and the Ricci scalar of $h_{\mu\nu}$ due to the fact that $g_{\mu\nu}$ is used here for the contraction of $\mathcal{R}_{\mu\nu}$.

f(R) gravity – Palatini approach

Replacing eqs. (26) and (27) in eq. (18), and after some easy manipulations, one obtains

$$
\begin{aligned}
G_{\mu\nu} &= \frac{\kappa}{f'} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(\mathcal{R} - \frac{f}{f'} \right) \\
&\quad + \frac{1}{f'} \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) f' - \\
&\quad - \frac{3}{2} \frac{1}{f'^2} \left[(\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2} g_{\mu\nu} (\nabla f')^2 \right] .
\end{aligned}
\tag{28}
$$

Various Theories



Conformal Transformation

$$R_{\alpha\beta\gamma}{}^{\delta} = \Gamma^{\delta}_{\alpha\gamma,\beta} - \Gamma^{\delta}_{\beta\gamma,\alpha} + \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\delta}_{\sigma\beta} - \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\delta}_{\sigma\alpha} \ ,$$

$$R_{\mu\rho} = \Gamma^{\nu}_{\mu\rho,\nu} - \Gamma^{\nu}_{\nu\rho,\mu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\nu}_{\alpha\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\nu}_{\alpha\mu} \ ,$$

and $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci curvature. $\Box \equiv g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ is d'Alembert's operator.

Conformal Transformation

$$R_{\alpha\beta\gamma}{}^{\delta} = \Gamma^{\delta}_{\alpha\gamma,\beta} - \Gamma^{\delta}_{\beta\gamma,\alpha} + \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\delta}_{\sigma\beta} - \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\delta}_{\sigma\alpha} \ ,$$

$$R_{\mu\rho} = \Gamma^{\nu}_{\mu\rho,\nu} - \Gamma^{\nu}_{\nu\rho,\mu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\nu}_{\alpha\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\nu}_{\alpha\mu} \ ,$$

and $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci curvature. $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is d'Alembert's operator.

If $(M, g_{\mu\nu})$ is a spacetime, the point–dependent rescaling of the metric tensor

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \ , \tag{1.1}$$

where $\Omega = \Omega(x)$ is a nonvanishing, regular function, is called a *Weyl* or *conformal transformation*. It affects the lengths of time [space]–like intervals and the norm of time [space]–like vectors, but it leaves the light cones unchanged: the spacetimes $(M, g_{\mu\nu})$ and $(M, \tilde{g}_{\mu\nu})$ have the same causal structure. The converse is also true (Wald 1984). If

Conformal Transformation

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \Omega^{-1}\left(\delta^{\alpha}_{\beta}\nabla_{\gamma}\Omega + \delta^{\alpha}_{\gamma}\nabla_{\beta}\Omega - g_{\beta\gamma}\nabla^{\alpha}\Omega\right) \ , \tag{1.2}$$

$$\begin{aligned}\widetilde{R_{\alpha\beta\gamma}}{}^{\delta} &= R_{\alpha\beta\gamma}{}^{\delta} + 2\delta^{\delta}_{[\alpha}\nabla_{\beta]}\nabla_{\gamma}(\ln\Omega) - 2g^{\delta\sigma}g_{\gamma[\alpha}\nabla_{\beta]}\nabla_{\sigma}(\ln\Omega) + 2\nabla_{[\alpha}(\ln\Omega)\delta^{\delta}_{\beta]}\nabla_{\gamma}(\ln\Omega) \\ &-2\nabla_{[\alpha}(\ln\Omega)g_{\beta]\gamma}g^{\delta\sigma}\nabla_{\sigma}(\ln\Omega) - 2g_{\gamma[\alpha}\delta^{\delta}_{\beta]}g^{\sigma\rho}\nabla_{\sigma}(\ln\Omega)\nabla_{\rho}(\ln\Omega) \ ,\end{aligned} \tag{1.3}$$

$$\begin{aligned}\tilde{R}_{\alpha\beta} &= R_{\alpha\beta} - (n-2)\nabla_{\alpha}\nabla_{\beta}(\ln\Omega) - g_{\alpha\beta}g^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}(\ln\Omega) + (n-2)\nabla_{\alpha}(\ln\Omega)\nabla_{\beta}(\ln\Omega) \\ &-(n-2)g_{\alpha\beta}\, g^{\rho\sigma}\nabla_{\rho}(\ln\Omega)\nabla_{\sigma}(\ln\Omega) \ ,\end{aligned} \tag{1.4}$$

$$\tilde{R} \equiv \tilde{g}^{\alpha\beta}\tilde{R}_{\alpha\beta} = \Omega^{-2}\left[R - 2\,(n-1)\,\Box\,(\ln\Omega) - (n-1)\,(n-2)\,\frac{g^{\alpha\beta}\nabla_{\alpha}\Omega\nabla_{\beta}\Omega}{\Omega^2}\right] \ , \tag{1.5}$$

where n ($n \geq 2$) is the dimension of the spacetime manifold M. In the case $n = 4$, the scalar curvature has the expressions

$$\tilde{R} = \Omega^{-2}\left[R - \frac{6\Box\Omega}{\Omega}\right] = \Omega^{-2}\left[R - \frac{12\Box(\sqrt{\Omega})}{\sqrt{\Omega}} - \frac{3g^{\alpha\beta}\nabla_{\alpha}\Omega\nabla_{\beta}\Omega}{\Omega^2}\right] \ , \tag{1.6}$$

Conformal Transformation

which are useful in many applications. The Weyl tensor ${C_{\alpha\beta\gamma}}^{\delta}$ (beware of the position of the indices !) is conformally invariant:

$$\widetilde{{C_{\alpha\beta\gamma}}^{\delta}} = {C_{\alpha\beta\gamma}}^{\delta} \; , \tag{1.7}$$

and the null geodesics are also conformally invariant (Lorentz 1937). The conservation equation $\nabla^{\nu} T_{\mu\nu} = 0$ for a symmetric stress–energy tensor $T_{\mu\nu}$ is not conformally invariant unless the trace $T \equiv {T^{\mu}}_{\mu}$ vanishes (Wald 1984). The Klein–Gordon equation $\Box\phi = 0$ for a scalar field ϕ is not conformally invariant, but its generalization

$$\Box\phi - \frac{n-2}{4(n-1)} R\,\phi = 0 \tag{1.8}$$

$(n \geq 2)$ is conformally invariant (note that the introduction of a nonzero cosmological constant in the Einstein action for gravity creates an effective mass, and a length scale,

Conformal Transformation of Brans Dicke theory

$$S_{BD} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi\, R - \frac{\omega}{\phi}\,\nabla^\mu\phi\nabla_\mu\phi\right] + S_{matter}\ , \tag{2.1}$$

which corresponds to the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right)\ , \tag{2.2}$$

$$\Box\Phi = \frac{8\pi T}{3+2\omega}\ . \tag{2.3}$$

The conformal transformation (1.1) with

$$\Omega = \sqrt{G\phi} \tag{2.4}$$

and the redefinition of the scalar field given in differential form by

$$d\tilde{\phi} = \sqrt{\frac{2\omega+3}{16\pi G}}\ \frac{d\phi}{\phi} \tag{2.5}$$

($\omega > -3/2$) transform the action (2.1) into the "Einstein frame" action

$$S = \int d^4x\left\{\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{\nabla}^\mu\tilde{\phi}\tilde{\nabla}_\mu\tilde{\phi}\right] + \exp\left(-8\sqrt{\frac{\pi G}{2\omega+3}}\tilde{\phi}\right)\mathcal{L}_{matter}(\tilde{g})\right\}\ , \tag{2.6}$$

where $\tilde{\nabla}_\mu$ is the covariant derivative operator of the rescaled metric $\tilde{g}_{\mu\nu}$. The gravita-

Conformal Transformation of non-minimal coupled theory

$$S = \int d^4x\sqrt{-g}\left[\left(\frac{1}{16\pi G} - \frac{\xi\phi^2}{2}\right)R - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right] , \tag{2.7}$$

where $V(\phi)$ is the scalar field potential (possibly including a mass term and the cosmological constant) and ξ is a dimensionless coupling constant. Note that the dimensions of the scalar field are $[\phi] = \left[G^{-1/2}\right] = [m_{pl}]$. The equation satisfied by the scalar ϕ is

$$\Box\phi - \xi R\phi - \frac{dV}{d\phi} = 0 . \tag{2.8}$$

Two cases occur most frequently in the literature: "minimal coupling" ($\xi = 0$) and "conformal coupling" ($\xi = 1/6$); the latter makes the wave equation (2.8) conformally invariant in four dimensions if $V = 0$ or $V = \lambda\phi^4$ (the latter potential being used in the chaotic inflationary scenario). The conformal transformation (1.1) with

$$\Omega^2 = 1 - 8\pi G\xi\phi^2 \tag{2.9}$$

and the redefinition of the scalar field, given in differential form by

$$d\tilde{\phi} = \frac{\left[1 - 8\pi G\xi\left(1 - 6\xi\right)\phi^2\right]^{1/2}}{1 - 8\pi G\xi\phi^2}\, d\phi , \tag{2.10}$$

Conformal Transformation of non-minimal coupled theory

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{\nabla}^{\mu}\tilde{\phi}\tilde{\nabla}_{\mu}\tilde{\phi} - \tilde{V}(\tilde{\phi})\right] , \qquad (2.11)$$

where the scalar field $\tilde{\phi}$ is now minimally coupled and satisfies the equation

$$\tilde{g}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\tilde{\phi} - \frac{d\tilde{V}}{d\tilde{\phi}} = 0 . \qquad (2.12)$$

The new scalar field potential is given by

$$\tilde{V}(\tilde{\phi}) = \frac{V(\phi)}{(1 - 8\pi G\xi\phi^2)^2} , \qquad (2.13)$$

Conformal Transformation of f(R) gravity

$$S_{met} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, f(R) + S_M(g_{\mu\nu}, \psi). \tag{50}$$

One can introduce a new field χ and write the dynamically equivalent action

$$\begin{aligned} S_{met} &= \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, [f(\chi) + f'(\chi)(R - \chi)] + \\ &\quad + S_M(g_{\mu\nu}, \psi). \end{aligned} \tag{51}$$

Variation with respect to χ leads to the equation

$$f''(\chi)(R - \chi) = 0. \tag{52}$$

Therefore, $\chi = R$ if $f''(\chi) \neq 0$, which reproduces the action (5).[12] Redefining the field χ by $\phi = f'(\chi)$ and setting

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)), \tag{53}$$

the action takes the form

$$S_{met} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, [\phi R - V(\phi)] + S_M(g_{\mu\nu}, \psi). \tag{54}$$

Conformal Transformation of f(R) gravity

The field equations corresponding to the action (54) are

$$G_{\mu\nu} = \frac{\kappa}{\phi} T_{\mu\nu} - \frac{1}{2\phi} g_{\mu\nu} V(\phi) + \frac{1}{\phi} \left(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi \right), \quad (55)$$

$$R = V'(\phi). \quad (56)$$

These field equations could have been derived directly from eq. (6) using the same field redefinitions that were mentioned above for the action. By taking the trace of eq. (55) in order to replace R in eq. (56), one gets

$$3\Box\phi + 2V(\phi) - \phi \frac{dV}{d\phi} = \kappa T. \quad (57)$$

This last equation determines the dynamics of ϕ for given matter sources.

Conformal Transformation of f(R) gravity

Finally, let us mention that, as usual in Brans–Dicke theory and more general scalar-tensor theories, one can perform a conformal transformation and rewrite the action (54) in what is called the Einstein frame (as opposed to the Jordan frame). Specifically, by performing the conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f'(R)\, g_{\mu\nu} \equiv \phi\, g_{\mu\nu} \tag{58}$$

and the scalar field redefinition $\phi = f'(R) \to \tilde{\phi}$ with

$$d\tilde{\phi} = \sqrt{\frac{2\omega_0 + 3}{2\kappa}}\, \frac{d\phi}{\phi}, \tag{59}$$

Conformal Transformation of f(R) gravity

a scalar-tensor theory is mapped into the Einstein frame in which the "new" scalar field $\tilde{\phi}$ couples minimally to the Ricci curvature and has canonical kinetic energy, as described by the gravitational action

$$S^{(g)} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2} \partial^\alpha \tilde{\phi} \partial_\alpha \tilde{\phi} - U(\tilde{\phi}) \right] . \tag{60}$$

For the $\omega_0 = 0$ equivalent of metric $f(R)$ gravity we have

$$\phi \equiv f'(R) = \mathrm{e}^{\sqrt{\frac{2\kappa}{3}}\,\tilde{\phi}}, \tag{61}$$

$$U(\tilde{\phi}) = \frac{R f'(R) - f(R)}{2\kappa \left(f'(R)\right)^2}, \tag{62}$$

where $R = R(\tilde{\phi})$, and the complete action is

$$\begin{aligned} S'_{met} \; &= \; \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2} \partial^\alpha \tilde{\phi} \partial_\alpha \tilde{\phi} - U(\tilde{\phi}) \right] + \\ & \quad + S_M (e^{-\sqrt{2\kappa/3}\,\tilde{\phi}} \tilde{g}_{\mu\nu}, \psi) . \end{aligned} \tag{63}$$

f(R) gravity

- $$\boxed{S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, f(R) + S_m\left(g_{\mu\nu},\psi\right)}$$

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \left[\nabla_\mu \nabla_\mu - g_{\mu\nu}\Diamond\right] f'(R) = 8\pi G\, T_{\mu\nu}$$

- Field Equations (metric formalism):

- Conformal transformation: $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = f'(R) g_{\mu\nu} \equiv \phi g_{\mu\nu},\ d\varphi = \sqrt{\frac{2\omega_0 + 3}{16\pi G}}\frac{d\phi}{\phi}$

$$\Rightarrow_{\omega_0=0}\ S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\partial^\alpha\varphi\partial_\alpha\varphi - U(\varphi)\right] + S_m\left(e^{-\sqrt{16\pi G/3}}\tilde{g}_{\mu\nu},\psi\right) \qquad U(\varphi) = \frac{Rf'(R) - f(R)}{16\pi G\left[f'(R)\right]^2}$$

Are the Jordan and Einstein frames equivalent?

- Are the Jordan and Einstein frames equivalent?

Are the Jordan and Einstein frames equivalent?

- Are the Jordan and Einstein frames equivalent?

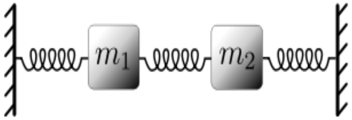
- Answers after >1000 papers of the literature:

- Yes
- No
- Yes at the classical level, no at the quantum level
- Yes only if matter is absent, no otherwise
- Yes only if singularities are absent, no otherwise

Hamiltonian Analysis – Degrees of Freedom

Coupled oscillators [edit]

Consider two equal bodies (not affected by gravity), each of mass m, attached to three springs, each with spring constant k. They are attached in the following manner, forming a system that is physically symmetric:



where the edge points are fixed and cannot move. We'll use $x_1(t)$ to denote the horizontal displacement of the left mass, and $x_2(t)$ to denote the displacement of the right mass.

If one denotes acceleration (the second derivative of $x(t)$ with respect to time) as $\ddot{x}$, the equations of motion are:

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -kx_2 + k(x_1 - x_2) = -2kx_2 + kx_1$$

Since we expect oscillatory motion of a normal mode (where ω is the same for both masses), we try:

$$x_1(t) = A_1 e^{i\omega t}$$
$$x_2(t) = A_2 e^{i\omega t}$$

Substituting these into the equations of motion gives us:

$$-\omega^2 m A_1 e^{i\omega t} = -2kA_1 e^{i\omega t} + kA_2 e^{i\omega t}$$
$$-\omega^2 m A_2 e^{i\omega t} = kA_1 e^{i\omega t} - 2kA_2 e^{i\omega t}$$

Hamiltonian Analysis – Degrees of Freedom

Since the exponential factor is common to all terms, we omit it and simplify:

$$(\omega^2 m - 2k)A_1 + kA_2 = 0$$
$$kA_1 + (\omega^2 m - 2k)A_2 = 0$$

And in matrix representation:

$$\begin{bmatrix} \omega^2 m - 2k & k \\ k & \omega^2 m - 2k \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

If the matrix on the left is invertible, the unique solution is the trivial solution $(A_1, A_2) = (x_1, x_2) = (0,0)$. The non trivial solutions are to be found for those values of ω whereby the matrix on the left is sin
i.e. is not invertible. It follows that the determinant of the matrix must be equal to 0, so:

$$(\omega^2 m - 2k)^2 - k^2 = 0$$

Solving for ω, we have two positive solutions:

$$\omega_1 = \sqrt{\frac{k}{m}}$$
$$\omega_2 = \sqrt{\frac{3k}{m}}$$

If we substitute ω_1 into the matrix and solve for (A_1, A_2), we get $(1, 1)$. If we substitute ω_2, we get $(1, -1)$. (These vectors are eigenvectors, and the frequencies are eigenvalues.)

The first normal mode is:

$$\vec{\eta}_1 = \begin{pmatrix} x_1^1(t) \\ x_2^1(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t + \varphi_1)$$

Which corresponds to both masses moving in the same direction at the same time. This mode is called antisymmetric.

The second normal mode is:

$$\vec{\eta}_2 = \begin{pmatrix} x_1^2(t) \\ x_2^2(t) \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t + \varphi_2)$$

Hamiltonian Analysis – Degrees of Freedom

$$T = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2}, \; V = \frac{k_1 x_1^2}{2} + \frac{k_2 (x_2 - x_1)^2}{2} + \frac{k_3 x_2^2}{2}.$$

The dots here (according to Newton's notation, which is widely used in mechanics and physics) refer to the first derivatives of the coordinates, i.e. velocities of the masses. The Lagrangian of the system is written as follows:

$$L = T - V = \frac{1}{2}\left[m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 - k_1 x_1^2 - k_2 (x_2 - x_1)^2 - k_3 x_2^2\right].$$

Compose the differential Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \quad \text{or} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1}, \; \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = \frac{\partial L}{\partial x_2}.$$

Hamiltonian Analysis – Degrees of Freedom

Find the partial derivatives:

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{1}{2} \cdot 2m_1\dot{x}_1 = m_1\dot{x}_1,$$
$$\frac{\partial L}{\partial x_1} = \frac{1}{2}\left[-2k_1x_1 + 2k_2\left(x_2 - x_1\right)\right] = -k_1x_1 + k_2\left(x_2 - x_1\right),$$

$$\frac{\partial L}{\partial \dot{x}_2} = \frac{1}{2} \cdot 2m_2\dot{x}_2 = m_2\dot{x}_2,$$
$$\frac{\partial L}{\partial x_2} = \frac{1}{2}\left[-2k_2\left(x_2 - x_1\right) - 2k_2x_2\right] = -k_2\left(x_2 - x_1\right) - k_3x_2.$$

Hamiltonian Analysis – Degrees of Freedom

$$\det\left(K+\omega^2 I\right)=0, \quad \Rightarrow \begin{vmatrix} -\frac{k_1+k_2}{m_1}+\omega^2 & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2}+\omega^2 \end{vmatrix}=0,$$

$$\Rightarrow \left(\omega^2-\frac{k_1+k_2}{m_1}\right)\cdot\left(\omega^2-\frac{k_2+k_3}{m_2}\right)-\frac{k_2^2}{m_1 m_2}=0,$$

$$\Rightarrow \omega^4-\frac{k_1+k_2}{m_1}\omega^2-\frac{k_2+k_3}{m_2}\omega^2+\frac{(k_1+k_2)(k_2+k_3)}{m_1 m_2}-\frac{k_2^2}{m_1 m_2}=0,$$

$$\Rightarrow \omega^4-\left(\frac{k_1+k_2}{m_1}+\frac{k_2+k_3}{m_2}\right)\omega^2+\frac{(k_1+k_2)(k_2+k_3)}{m_1 m_2}-\frac{k_2^2}{m_1 m_2}=0.$$

Solving this biquadratic equation, we find the eigenfrequencies. Let us first compute the discriminant:

$$D=\left(\frac{k_1+k_2}{m_1}+\frac{k_2+k_3}{m_2}\right)^2-4\left[\frac{(k_1+k_2)(k_2+k_3)}{m_1 m_2}-\frac{k_2^2}{m_1 m_2}\right]$$

$$=\left(\frac{k_1+k_2}{m_1}\right)^2+\left(\frac{k_2+k_3}{m_2}\right)^2+\frac{2(k_1+k_2)(k_2+k_3)}{m_1 m_2}-\frac{4(k_1+k_2)(k_2+k_3)}{m_1 m_2}$$

$$+\frac{4k_2^2}{m_1 m_2}=\left(\frac{k_1+k_2}{m_1}-\frac{k_2+k_3}{m_2}\right)^2+\frac{4k_2^2}{m_1 m_2}.$$

Then the square of the eigenfrequencies will be described by the formula

$$\omega^2=\frac{1}{2}\left\{\left(\frac{k_1+k_2}{m_1}+\frac{k_2+k_3}{m_2}\right)\pm\left[\left(\frac{k_1+k_2}{m_1}-\frac{k_2+k_3}{m_2}\right)^2+\frac{4k_2^2}{m_1 m_2}\right]^{\frac{1}{2}}\right\}$$

Hamiltonian Analysis – Degrees of Freedom

Calculating a Hamiltonian from a Lagrangian [edit]

Given a Lagrangian in terms of the generalized coordinates q^i and generalized velocities $\dot{q}^i$ and time,

1. The momenta are calculated by differentiating the Lagrangian with respect to the (generalized) velocities:
$$p_i(q^i, \dot{q}^i, t) = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$$
2. The velocities $\dot{q}^i$ are expressed in terms of the momenta p_i by inverting the expressions in the previous step.
3. The Hamiltonian is calculated using the usual definition of $\mathcal{H}$ as the Legendre transformation of $\mathcal{L}$:
$$\mathcal{H} = \sum_i \dot{q}^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L} = \sum_i \dot{q}^i p_i - \mathcal{L}$$
Then the velocities are substituted for through the above results.

Hamiltonian Analysis – Degrees of Freedom

The four-dimensional line element in the ADM form is given by,

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -(N^2 - N_i N^i) d\eta^2 + 2N_i dx^i d\eta + \gamma_{ij} dx^i dx^j , \end{aligned} \tag{2.6}$$

where $N(x^\mu)$ and $N_i(x^\mu)$ are Lapse function and Shift vector respectively, γ_{ij} is the 3-D space

Hamiltonian Analysis – Degrees of Freedom – GR case

2.2 The ADM formulation of general relativity

The physical content of gravity and its propagating modes are easily identified in the ADM formulation [22] which is based on a $3+1$ decomposition of the metric,

$$N \equiv (-g^{00})^{-1/2}\,, \qquad N_i \equiv g_{0i}\,, \qquad \gamma_{ij} \equiv g_{ij}\,. \tag{2.4}$$

The N and N_i are the lapse and shift functions respectively. In this parameterization,

$$g^{\mu\nu} = N^{-2} \begin{pmatrix} -1 & N^j \\ N^i & N^2\gamma^{ij} - N^i N^j \end{pmatrix}, \tag{2.5}$$

where, $N^j = \gamma^{jk} N_k$ and $\gamma^{ij}\gamma_{jk} = \delta^i_k$. Denoting the momentum canonically conjugate to γ_{ij} by π^{ij}, the Einstein-Hilbert action in terms of these variables becomes (we ignore all boundary terms in what follows),

$$S = M_p^2 \int d^4x \left[\pi^{ij}\partial_t\gamma_{ij} + N R^0 + N_i R^i\right]\,. \tag{2.6}$$

The R^μ are functions of γ_{ij} and π^{ij} but are independent of the $N_\mu = (N, N_i)$,

$$R^0 = \sqrt{\det\gamma}\left[R(\gamma) + \frac{1}{\det\gamma}(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij})\right], \qquad R^i = 2\sqrt{\det\gamma}\,\nabla_j\left(\frac{\pi^{ij}}{\sqrt{\det\gamma}}\right). \tag{2.7}$$

Hamiltonian Analysis – Degrees of Freedom – GR case

(2.7)

The six components of γ_{ij} are potentially propagating modes in the sense that their equations of motion obtained from (2.6), as well as those for their conjugate momenta π^{ij}, involve time derivatives (so that the Euler-Lagrange equations for γ_{ij} are second order in time). Since a single propagating mode involves a field component and its canonically conjugate momentum, the six potentially propagating modes are described by the 12 functions (γ_{ij}, π^{ij}). However, in the theory defined by (2.6) not all of these are independent. To see this, note that the N_μ appear linearly as Lagrange multipliers, hence their equations of motion are four constraints (the "Hamiltonian" and "momentum" constraints) on the remaining fields,

$$R_0(\gamma, \pi) = 0\,, \qquad R_i(\gamma, \pi) = 0\,. \tag{2.8}$$

These constraints can be used along with the four general coordinate transformations to eliminate eight of the 12 functions, in favor of two remaining pairs. These pairs are the two propagating modes of GR, describing the two polarization states of the massless graviton at the non-linear level. In particular, the scalar ghost is not part of the physical spectrum [22].

Of the 12 equations of motion for (γ_{ij}, π^{ij}), four reduce to Bianchi identities while another four determine the N_μ. The remaining four equations describe the propagating modes.

Hamiltonian Analysis – Degrees of Freedom – f(R) case

Let us consider now the Lagrangian density describing a generic $f(R)$ model, namely

$$\mathcal{L} = \sqrt{-g}\big(f(R) - 2\Lambda_c\big), \qquad \text{with } f'' \neq 0. \tag{65}$$

A cosmological term is added also in this case for the sake of generality. Obviously $f'' = 0$ corresponds to General Relativity. The generalized Hamiltonian density for the $f(R)$ theory assumes the form

$$\mathcal{H} = \frac{1}{2\kappa}\left[\frac{\mathcal{P}}{6}\left({}^{(3)}R - 2\Lambda_c - 3K_{ij}K^{ij} + K^2\right) + V(\mathcal{P}) - \frac{1}{3}\,g^{ij}\mathcal{P}_{|ij} - 2p^{ij}K_{ij}\right], \tag{66}$$

where

$$V(\mathcal{P}) = \sqrt{-g}\big[Rf'(R) - f(R)\big]. \tag{67}$$

Henceforth, the superscript 3 indicating the spatial part of the metric will be omitted on the metric itself. When $f(R) = R$, $V(\mathcal{P}) = 0$ as it should be. Since

$$\mathcal{P}^{ij} = -2\sqrt{-g}g^{ij}f'(R) \qquad \Longrightarrow \qquad \mathcal{P} = -\,6\sqrt{-g}f'(R), \tag{68}$$

we have

$$\mathcal{H} = \frac{1}{2\kappa}\left[-\sqrt{-g}f'(R)\left({}^{(3)}R - 2\Lambda_c - 3K_{ij}K^{ij} + K^2\right) + V(\mathcal{P}) + 2g^{ij}\left(\sqrt{-g}f'(R)\right)_{|ij} - 2p^{ij}K_{ij}\right]. \tag{69}$$

Hamiltonian Analysis – Degrees of Freedom – f(R) case

With the help of Eq. (57), Eq.(83) becomes

$$\mathcal{H} = f'(R)\left[(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{-g}}{2\kappa}\left({}^{(3)}R - 2\Lambda_c\right)\right] + \frac{1}{2\kappa}\left[\sqrt{-g}f'(R)\left(2K_{ij}K^{ij}\right) + V(\mathcal{P}) + 2g^{ij}\left(\sqrt{-g}f'(R)\right)_{|ij} - 2p^{ij}K_{ij}\right]. \quad (70)$$

However

$$p^{ij} = \sqrt{-g}K^{ij}, \quad (71)$$

then we obtain

$$\mathcal{H} = f'(R)\left[(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{-g}}{2\kappa}\left({}^{(3)}R - 2\Lambda_c\right)\right] + \frac{1}{2\kappa}\left[2\sqrt{-g}K_{ij}K^{ij}\left(f'(R) - 1\right) + V(\mathcal{P}) + 2g^{ij}\left(\sqrt{-g}f'(R)\right)_{|ij}\right] \quad (72)$$

and transforming into canonical momenta, one gets

$$\mathcal{H} = f'(R)\left[(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{-g}}{2\kappa}\left({}^{(3)}R - 2\Lambda_c\right)\right] + 2(2\kappa)\left[G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4}\right]\left(f'(R) - 1\right) + \frac{1}{2\kappa}\left[V(\mathcal{P}) + 2g^{ij}\left(\sqrt{-g}f'(R)\right)_{|ij}\right]. \quad (73$$

By imposing the Hamiltonian constraint, we obtain

$$f'(R)\left[(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{-g}}{2\kappa}{}^{(3)}R\right] + 2(2\kappa)\left[G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4}\right]\left(f'(R) - 1\right) + \frac{1}{2\kappa}\left[V(\mathcal{P}) + 2g^{ij}\left(\sqrt{-g}f'(R)\right)_{|ij}\right] = -f'(R)\sqrt{-g}\,\frac{\Lambda_c}{\kappa}$$

Hamiltonian Analysis – Degrees of Freedom

- In summary:

- General Relativity has 2 propagating d.o.f, corresponding to a massless spin-2 field (graviton)

- f(R) gravity has 2+1 propagating d.o.f.

- This extra degree of freedom brings new and interesting phenomenology in cosmology, black holes, etc

R^2 cosmology – Inflation – Jordan Frame

We consider the models of the form

$$f(R) = R + \alpha R^n \,, \qquad (\alpha > 0, n > 0) \,, \tag{3.1}$$

which include the Starobinsky's model [564] as a specific case ($n = 2$). In the absence of the matter fluid ($\rho_M = 0$), Eq. (2.15) gives

$$3(1 + n\alpha R^{n-1})H^2 = \frac{1}{2}(n-1)\alpha R^n - 3n(n-1)\alpha H R^{n-2}\dot{R} \,. \tag{3.2}$$

The cosmic acceleration can be realized in the regime $F = 1 + n\alpha R^{n-1} \gg 1$. Under the approximation $F \simeq n\alpha R^{n-1}$, we divide Eq. (3.2) by $3n\alpha R^{n-1}$ to give

$$H^2 \simeq \frac{n-1}{6n}\left(R - 6nH\frac{\dot{R}}{R}\right) \,. \tag{3.3}$$

R^2 cosmology – Inflation – Jordan Frame

During inflation the Hubble parameter H evolves slowly so that one can use the approximation $|\dot{H}/H^2| \ll 1$ and $|\ddot{H}/(H\dot{H})| \ll 1$. Then Eq. (3.3) reduces to

$$\frac{\dot{H}}{H^2} \simeq -\epsilon_1, \qquad \epsilon_1 = \frac{2-n}{(n-1)(2n-1)}\,. \tag{3.4}$$

Integrating this equation for $\epsilon_1 > 0$, we obtain the solution

$$H \simeq \frac{1}{\epsilon_1 t}, \qquad a \propto t^{1/\epsilon_1}\,. \tag{3.5}$$

The cosmic acceleration occurs for $\epsilon_1 < 1$, i.e., $n > (1+\sqrt{3})/2$. When $n = 2$ one has $\epsilon_1 = 0$, so that H is constant in the regime $F \gg 1$. The models with $n > 2$ lead to super inflation characterized by $\dot{H} > 0$ and $a \propto |t_0 - t|^{-1/|\epsilon_1|}$ (t_0 is a constant). Hence the standard inflation with decreasing H occurs for $(1+\sqrt{3})/2 < n < 2$.

In the following let us focus on the Starobinsky's model given by

$$f(R) = R + R^2/(6M^2)\,, \tag{3.6}$$

R^2 cosmology – Inflation – Jordan Frame

where the constant M has a dimension of mass. The presence of the linear term in R eventually causes inflation to end. Without neglecting this linear term, the combination of Eqs. (2.15) and (2.16) gives

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{1}{2}M^2 H = -3H\dot{H}\,, \tag{3.7}$$

$$\ddot{R} + 3H\dot{R} + M^2 R = 0\,. \tag{3.8}$$

R^2 cosmology – Inflation – Jordan Frame

During inflation the first two terms in Eq. (3.7) can be neglected relative to others, which gives $\dot{H} \simeq -M^2/6$. We then obtain the solution

$$
\begin{aligned}
H &\simeq H_i - (M^2/6)(t - t_i)\,, && (3.9)\\
a &\simeq a_i \exp\left[H_i(t - t_i) - (M^2/12)(t - t_i)^2\right]\,, && (3.10)\\
R &\simeq 12H^2 - M^2\,, && (3.11)
\end{aligned}
$$

where H_i and a_i are the Hubble parameter and the scale factor at the onset of inflation ($t = t_i$), respectively. This inflationary solution is a transient attractor of the dynamical system [407]. The accelerated expansion continues as long as the slow-roll parameter

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \simeq \frac{M^2}{6H^2}\,, \qquad (3.12)$$

is smaller than the order of unity, i.e., $H^2 \gtrsim M^2$. One can also check that the approximate relation $3H\dot{R} + M^2 R \simeq 0$ holds in Eq. (3.8) by using $R \simeq 12H^2$. The end of inflation (at time $t = t_f$) is characterized by the condition $\epsilon_f \simeq 1$, i.e., $H_f \simeq M/\sqrt{6}$. From Eq. (3.11) this corresponds to the epoch at which the Ricci scalar decreases to $R \simeq M^2$. As we will see later, the WMAP normalization of the CMB temperature anisotropies constrains the mass scale to be $M \simeq 10^{13}$ GeV. Note that the phase space analysis for the model (3.6) was carried out in [407, 24, 131].

R^2 cosmology – Inflation – Jordan Frame

We define the number of e-foldings from $t = t_i$ to $t = t_f$:

$$N \equiv \int_{t_i}^{t_f} H \, \mathrm{d}t \simeq H_i(t_f - t_i) - \frac{M^2}{12}(t_f - t_i)^2 \,. \tag{3.13}$$

Since inflation ends at $t_f \simeq t_i + 6H_i/M^2$, it follows that

$$N \simeq \frac{3H_i^2}{M^2} \simeq \frac{1}{2\epsilon_1(t_i)} \,, \tag{3.14}$$

where we used Eq. (3.12) in the last approximate equality. In order to solve horizon and flatness problems of the big bang cosmology we require that $N \gtrsim 70$ [391], i.e., $\epsilon_1(t_i) \lesssim 7 \times 10^{-3}$. The CMB temperature anisotropies correspond to the perturbations whose wavelengths crossed the Hubble radius around $N = 55-60$ before the end of inflation.

R^2 cosmology – Inflation – Einstein Frame

Dynamics in the Einstein frame

Let us consider inflationary dynamics in the Einstein frame for the model ($\boxed{3.6}$) in the absence of matter fluids ($\mathcal{L}_M = 0$). The action in the Einstein frame corresponds to ($\boxed{2.32}$) with a field ϕ defined by

$$\phi = \sqrt{\frac{3}{2}}\frac{1}{\kappa}\ln F = \sqrt{\frac{3}{2}}\frac{1}{\kappa}\ln\left(1 + \frac{R}{3M^2}\right) . \tag{3.15}$$

R^2 cosmology – Inflation – Einstein Frame

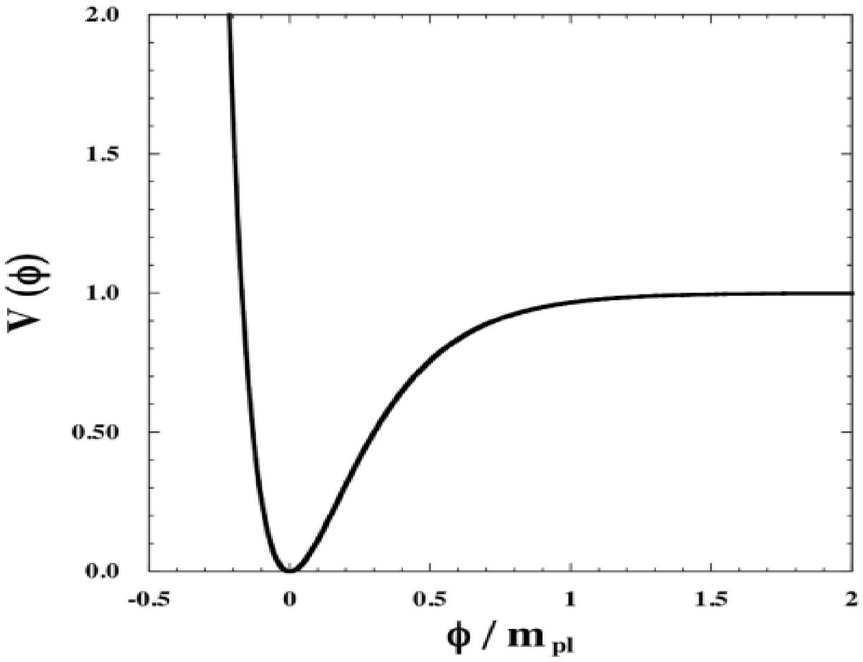


Figure 1: The field potential (3.16) in the Einstein frame corresponding to the model (3.6). Inflation is realized in the regime $\kappa\phi \gg 1$.

Using this relation, the field potential (2.33) reads [408, 61, 63]

$$V(\phi) = \frac{3M^2}{4\kappa^2}\left(1 - e^{-\sqrt{2/3}\kappa\phi}\right)^2 . \tag{3.16}$$

R^2 cosmology – Inflation – Einstein Frame

Since $F \simeq 4H^2/M^2$ during inflation, the transformation (2.44) gives a relation between the cosmic time $\tilde{t}$ in the Einstein frame and that in the Jordan frame:

$$\tilde{t} = \int_{t_i}^{t} \sqrt{F}\,\mathrm{d}t \simeq \frac{2}{M}\left[H_i(t-t_i) - \frac{M^2}{12}(t-t_i)^2\right], \tag{3.18}$$

where $t = t_i$ corresponds to $\tilde{t} = 0$. The end of inflation ($t_f \simeq t_i + 6H_i/M^2$) corresponds to $\tilde{t}_f = (2/M)N$ in the Einstein frame, where N is given in Eq. (3.13). On using Eqs. (3.10) and (3.18), the scale factor $\tilde{a} = \sqrt{F}a$ in the Einstein frame evolves as

$$\tilde{a}(\tilde{t}) \simeq \left(1 - \frac{M^2}{12H_i^2}M\tilde{t}\right)\tilde{a}_i\, e^{M\tilde{t}/2}, \tag{3.19}$$

where $\tilde{a}_i = 2H_i a_i/M$. Similarly the evolution of the Hubble parameter $\tilde{H} = (H/\sqrt{F})[1+\dot{F}/(2HF)]$ is given by

$$\tilde{H}(\tilde{t}) \simeq \frac{M}{2}\left[1 - \frac{M^2}{6H_i^2}\left(1 - \frac{M^2}{12H_i^2}M\tilde{t}\right)^{-2}\right], \tag{3.20}$$

R^2 cosmology – Inflation – Einstein Frame

The field equations for the action (2.32) are given by

$$3\tilde{H}^2 = \kappa^2 \left[\frac{1}{2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tilde{t}} \right)^2 + V(\phi) \right], \tag{3.21}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tilde{t}^2} + 3\tilde{H}\frac{\mathrm{d}\phi}{\mathrm{d}\tilde{t}} + V_{,\phi} = 0\,. \tag{3.22}$$

Using the slow-roll approximations $(\mathrm{d}\phi/\mathrm{d}\tilde{t})^2 \ll V(\phi)$ and $|\mathrm{d}^2\phi/\mathrm{d}\tilde{t}^2| \ll |\tilde{H}\mathrm{d}\phi/\mathrm{d}\tilde{t}|$ during inflation, one has $3\tilde{H}^2 \simeq \kappa^2 V(\phi)$ and $3\tilde{H}(\mathrm{d}\phi/\mathrm{d}\tilde{t}) + V_{,\phi} \simeq 0$. We define the slow-roll parameters

$$\tilde{\epsilon}_1 \equiv -\frac{\mathrm{d}\tilde{H}/\mathrm{d}\tilde{t}}{\tilde{H}^2} \simeq \frac{1}{2\kappa^2}\left(\frac{V_{,\phi}}{V}\right)^2, \qquad \tilde{\epsilon}_2 \equiv \frac{\mathrm{d}^2\phi/\mathrm{d}\tilde{t}^2}{\tilde{H}(\mathrm{d}\phi/\mathrm{d}\tilde{t})} \simeq \tilde{\epsilon}_1 - \frac{V_{,\phi\phi}}{3\tilde{H}^2}\,. \tag{3.23}$$

R^2 cosmology – Inflation – Einstein Frame

For the potential (3.16) it follows that

$$\tilde{\epsilon}_1 \simeq \frac{4}{3}(e^{\sqrt{2/3}\kappa\phi} - 1)^{-2}\,, \qquad \tilde{\epsilon}_2 \simeq \tilde{\epsilon}_1 + \frac{M^2}{3\tilde{H}^2}e^{-\sqrt{2/3}\kappa\phi}(1 - 2e^{-\sqrt{2/3}\kappa\phi})\,, \tag{3.24}$$

which are much smaller than 1 during inflation ($\kappa\phi \gg 1$). The end of inflation is characterized by the condition $\{\tilde{\epsilon}_1, |\tilde{\epsilon}_2|\} = \mathcal{O}(1)$. Solving $\tilde{\epsilon}_1 = 1$, we obtain the field value $\phi_f \simeq 0.19\, m_{\rm pl}$.

We define the number of e-foldings in the Einstein frame,

$$\tilde{N} = \int_{\tilde{t}_i}^{\tilde{t}_f} \tilde{H}\,{\rm d}\tilde{t} \simeq \kappa^2 \int_{\phi_f}^{\phi_i} \frac{V}{V_{,\phi}}{\rm d}\phi\,, \tag{3.25}$$

where ϕ_i is the field value at the onset of inflation. Since $\tilde{H}{\rm d}\tilde{t} = H{\rm d}t[1 + \dot{F}/(2HF)]$, it follows that $\tilde{N}$ is identical to N in the slow-roll limit: $|\dot{F}/(2HF)| \simeq |\dot{H}/H^2| \ll 1$. Under the condition $\kappa\phi_i \gg 1$ we have

$$\tilde{N} \simeq \frac{3}{4}e^{\sqrt{2/3}\kappa\phi_i}\,. \tag{3.26}$$

This shows that $\phi_i \simeq 1.11 m_{\rm pl}$ for $\tilde{N} = 70$. From Eqs. (3.24) and (3.26) together with the approximate relation $\tilde{H} \simeq M/2$, we obtain

$$\tilde{\epsilon}_1 \simeq \frac{3}{4\tilde{N}^2}\,, \qquad \tilde{\epsilon}_2 \simeq \frac{1}{\tilde{N}}\,, \tag{3.27}$$

where, in the expression of $\tilde{\epsilon}_2$, we have dropped the terms of the order of $1/\tilde{N}^2$. The results (3.27) will be used to estimate the spectra of density perturbations in Section 7.

R^2 cosmology – Inflation – Einstein Frame

- Firedmann Equations (metric formalism):

$$3FH^2 = \frac{FR-f}{2} - 3H\dot{F} + 8\pi G\,\rho_m$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} + 8\pi G\left(\rho_m + p_m\right)$$

$$F(R) \equiv f'(R)$$

$$R = 12H^2 + 6\dot{H}$$

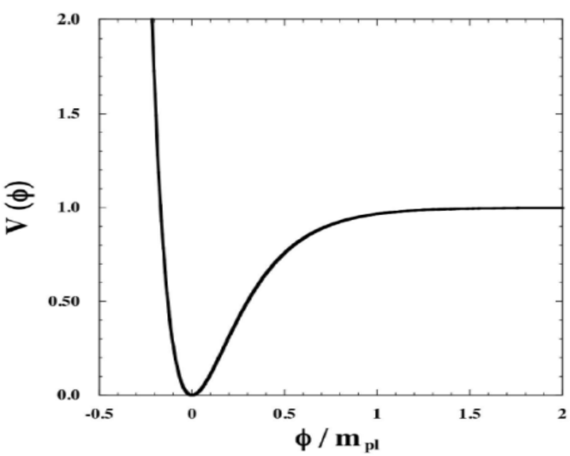
- Inflation: e.g. Starobinsky inflation

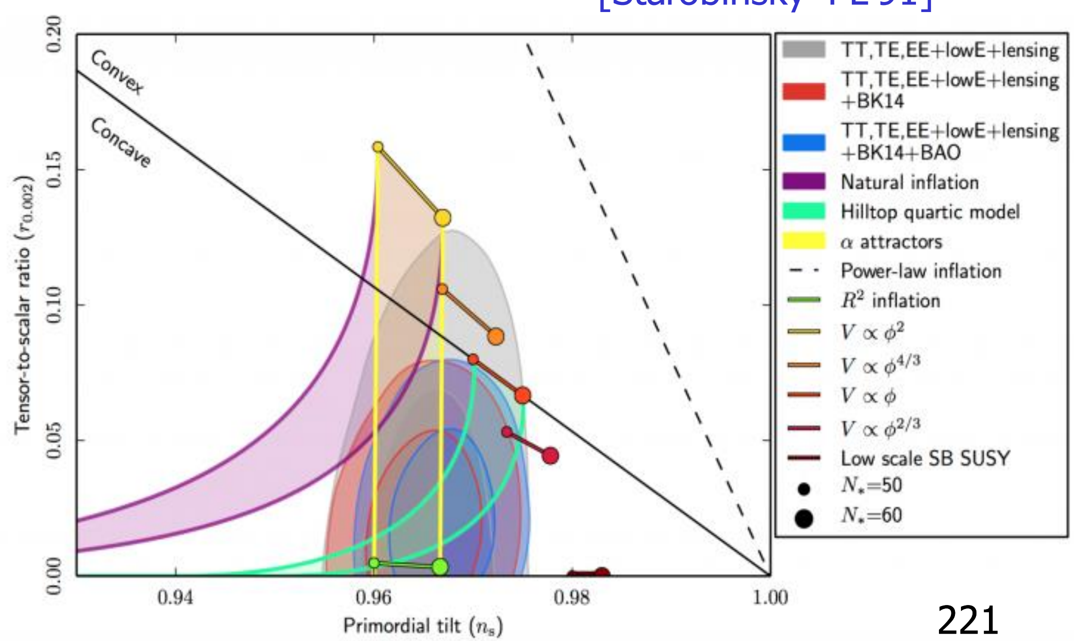
$$\boxed{f(R) = R + \frac{R^2}{6M^2}} \qquad \Rightarrow V(\phi) = \frac{3M^2}{32\pi G}\left(1 - e^{-\sqrt{2/3}8\pi G\phi}\right)$$

[Starobinsky PL 91]

$$H \approx H_i - \frac{M^2}{6}(t - t_i)$$

$$T_{reh} \le 3\times10^{17} g_*^{1/4}\left(\frac{M}{m_{pl}}\right)^{3/2} GeV \qquad M \approx 3\times10^{13}\ GeV$$





f(R) gravity - perturbations

$$
\begin{aligned}
&F R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu F + g_{\mu\nu} \Box F \\
&= \kappa^2 \left[\omega \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \phi \nabla_\lambda \phi \right) - V g_{\mu\nu} + T_{\mu\nu}^{(M)} \right], \qquad (6.3) \\
&\Box \phi + \frac{1}{2\omega} \left(\omega_{,\phi} \nabla^\lambda \phi \nabla_\lambda \phi - 2 V_{,\phi} + \frac{f_{,\phi}}{\kappa^2} \right) = 0 \,, \qquad (6.4)
\end{aligned}
$$

$$
ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)
$$

For the action (6.2) the background equations (without metric perturbations) are given by

$$
\begin{aligned}
&3FH^2 = \frac{1}{2}(RF - f) - 3H\dot{F} + \kappa^2 \left[\frac{1}{2} \omega \dot{\phi}^2 + V(\phi) + \rho_M \right], \qquad (6.6) \\
&-2F\dot{H} = \ddot{F} - H\dot{F} + \kappa^2 \omega \dot{\phi}^2 + \kappa^2 (\rho_M + P_M) \,, \qquad (6.7) \\
&\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega} \left(\omega_{,\phi} \dot{\phi}^2 + 2V_{,\phi} - \frac{f_{,\phi}}{\kappa^2} \right) = 0 \,, \qquad (6.8) \\
&\dot{\rho}_M + 3H(\rho_M + P_M) = 0 \,, \qquad (6.9)
\end{aligned}
$$

f(R) gravity - perturbations

We start with a general perturbed metric about the flat FLRW background [57, 352, 231, 232, 437]

$$ds^2 = -(1+2\alpha)\,dt^2 - 2a(t)\,(\partial_i\beta - S_i)dt\,dx^i + a^2(t)(\delta_{ij} + 2\psi\delta_{ij} + 2\partial_i\partial_j\gamma + 2\partial_j F_i + h_{ij})\,dx^i\,dx^j\,, \quad (6.1)$$

where α, β, ψ, γ are scalar perturbations, S_i, F_i are vector perturbations, and h_{ij} is the tensor perturbations, respectively. In this review we focus on scalar and tensor perturbations, because vector perturbations are generally unimportant in cosmology [71].

f(R) gravity - perturbations

We start with a general perturbed metric about the flat FLRW background [57, 352, 231, 232, 437]

$$ds^2 = -(1+2\alpha)\,dt^2 - 2a(t)\,(\partial_i\beta - S_i)dt\,dx^i + a^2(t)(\delta_{ij} + 2\psi\delta_{ij} + 2\partial_i\partial_j\gamma + 2\partial_j F_i + h_{ij})\,dx^i\,dx^j\,, \quad (6.1)$$

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where $T^{(M)}_{\mu\nu}$ is the energy-momentum tensor of matter.

We decompose ϕ and F into homogeneous and perturbed parts, $\phi = \bar{\phi} + \delta\phi$ and $F = \bar{F} + \delta F$, respectively. In the following we omit the bar for simplicity. The energy-momentum tensor of an ideal fluid with perturbations is

$$T^0_0 = -(\rho_M + \delta\rho_M)\,, \quad T^0_i = -(\rho_M + P_M)\partial_i v\,, \quad T^i_j = (P_M + \delta P_M)\delta^i_j\,, \quad (6.5)$$

where v characterizes the velocity potential of the fluid. The conservation of the energy-momentum tensor ($\nabla^\mu T_{\mu\nu} = 0$) holds for the theories with the action (6.2) [357].

f(R) gravity - perturbations

Perturbing Einstein equations at linear order, we obtain the following equations [315, 316] (see also [436, 566, 355, 438, 311, 312, 313, 492, 138, 33, 441, 328])

$$\frac{\Delta}{a^2}\psi + HA = -\frac{1}{2F}\left[\left(3H^2 + 3\dot{H} + \frac{\Delta}{a^2}\right)\delta F - 3H\dot{\delta F} + \frac{1}{2}\left(\kappa^2\omega_{,\phi}\dot{\phi}^2 + 2\kappa^2 V_{,\phi} - f_{,\phi}\right)\delta\phi \right.$$
$$\left. + \kappa^2\omega\dot{\phi}\dot{\delta\phi} + (3H\dot{F} - \kappa^2\omega\dot{\phi}^2)\alpha + \dot{F}A + \kappa^2\delta\rho_M\right], \qquad (6.11)$$

$$H\alpha - \dot{\psi} = \frac{1}{2F}\left[\kappa^2\omega\dot{\phi}\delta\phi + \dot{\delta F} - H\delta F - \dot{F}\alpha + \kappa^2(\rho_M + P_M)v\right], \qquad (6.12)$$

$$\dot{\chi} + H\chi - \alpha - \psi = \frac{1}{F}(\delta F - \dot{F}\chi), \qquad (6.13)$$

$$\dot{A} + 2HA + \left(3H + \frac{\Delta}{a^2}\right)\alpha = \frac{1}{2F}\left[3\ddot{\delta F} + 3H\dot{\delta F} - \left(6H^2 + \frac{\Delta}{a^2}\right)\delta F + 4\kappa^2\omega\dot{\phi}\dot{\delta\phi} \right.$$
$$+ (2\kappa^2\omega_{,\phi}\dot{\phi}^2 - 2\kappa^2 V_{,\phi} + f_{,\phi})\delta\phi - 3\dot{F}\dot{\alpha} - \dot{F}A$$
$$\left. - (4\kappa^2\omega\dot{\phi}^2 + 3H\dot{F} + 6\ddot{F})\alpha + \kappa^2(\delta\rho_M + \delta P_M)\right], \qquad (6.14)$$

f(R) gravity - perturbations

For later convenience, we define the following perturbed quantities

$$\chi \equiv a(\beta + a\dot{\gamma}), \qquad A \equiv 3(H\alpha - \dot{\psi}) - \frac{\Delta}{a^2}\chi. \tag{6.10}$$

$$\delta\ddot{F} + 3H\delta\dot{F} - \left(\frac{\Delta}{a^2} + \frac{R}{3}\right)\delta F + \frac{2}{3}\kappa^2\dot{\phi}\delta\dot{\phi} + \frac{1}{3}(\kappa^2\omega_{,\phi}\dot{\phi}^2 - 4\kappa^2 V_{,\phi} + 2f_{,\phi})\delta\phi$$
$$= \frac{1}{3}\kappa^2(\delta\rho_M - 3\delta P_M) + \dot{F}(A + \dot{\alpha}) + \left(2\ddot{F} + 3H\dot{F} + \frac{2}{3}\kappa^2\omega\dot{\phi}^2\right)\alpha - \frac{1}{3}F\delta R, \tag{6.15}$$

$$\delta\ddot{\phi} + \left(3H + \frac{\omega_{,\phi}}{\omega}\dot{\phi}\right)\delta\dot{\phi} + \left[-\frac{\Delta}{a^2} + \left(\frac{\omega_{,\phi}}{\omega}\right)_{,\phi}\frac{\dot{\phi}^2}{2} + \left(\frac{2V_{,\phi} - f_{,\phi}}{2\omega}\right)_{,\phi}\right]\delta\phi$$
$$= \dot{\phi}\dot{\alpha} + \left(2\ddot{\phi} + 3H\dot{\phi} + \frac{\omega_{,\phi}}{\omega}\dot{\phi}^2\right)\alpha + \dot{\phi}A + \frac{1}{2\omega}F_{,\phi}\delta R, \tag{6.16}$$

$$\delta\dot{\rho}_M + 3H(\delta\rho_M + \delta P_M) = (\rho_M + P_M)\left(A - 3H\alpha + \frac{\Delta}{a^2}v\right), \tag{6.17}$$

$$\frac{1}{a^3(\rho_M + P_M)}\frac{\mathrm{d}}{\mathrm{d}t}[a^3(\rho_M + P_M)v] = \alpha + \frac{\delta P_M}{\rho_M + P_M}, \tag{6.18}$$

where δR is given by

$$\delta R = -2\left[\dot{A} + 4HA + \left(\frac{\Delta}{a^2} + 3\dot{H}\right)\alpha + 2\frac{\Delta}{a^2}\psi\right]. \tag{6.19}$$

$$\boxed{G_{eff} = \frac{G}{f'}\,\frac{1 + 4\dfrac{k^2}{a^2}\dfrac{f''}{f'}}{1 + 3\dfrac{k^2}{a^2}\dfrac{f''}{f'}}}$$

f(R) cosmology – Dark energy

$$8\pi G\,\rho_{DE} = \frac{FR-f}{2} - 3H\dot{F} + 3H^2(1-F)$$

$$8\pi G\,p_{DE} = \ddot{F} + 2H\dot{F} - \frac{FR-f}{2} - \left(3H^2 + 2\dot{H}\right)(1-F)$$

for viable: $f_{,R} > 0,\ f_{,RR} > 0,\ for\ R \geq R_0(>0)$

Matter perturbations in viable f(R) models

Under the sub-horizon approximation ($k \gg aH$, k is a comoving wavenumber), the matter perturbation δ_m satisfies

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\rm eff}\rho_m\delta_m \simeq 0$$

where $G_{\rm eff} = \frac{G}{F}\frac{1+4\frac{k^2}{a^2R}m}{1+3\frac{k^2}{a^2R}m}$ $\quad F = \frac{\partial f}{\partial R}$

$m = Rf_{,RR}/f_{,R}$ is the deviation parameter from the LCDM.

$(m = 0 \quad \text{for} \quad f = R - 2\Lambda)$

f(R) cosmology – Dark energy

$$8\pi G\,\rho_{DE} = \frac{FR-f}{2} - 3H\dot{F} + 3H^2(1-F)$$

$$8\pi G\,p_{DE} = \ddot{F} + 2H\dot{F} - \frac{FR-f}{2} - \left(3H^2 + 2\dot{H}\right)(1-F)$$

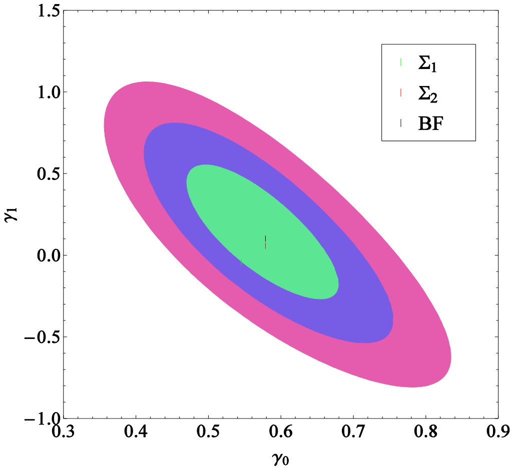
for viable: $f_{,R} > 0,\ f_{,RR} > 0,\ \text{for } R \geq R_0(>0)$

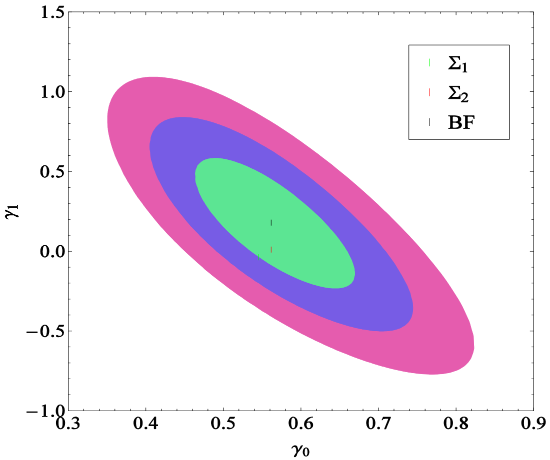
model	$f(R)$	Constant parameters
(i) Hu-Sawicki	$R - \frac{c_1 R_{\rm HS}(R/R_{\rm HS})^p}{c_2(R/R_{\rm HS})^p+1}$	$c_1,\ c_2,\ p(>0), R_{\rm HS}(>0)$
(ii) Starobinsky	$R + \lambda R_{\rm S}\left[\left(1+\frac{R^2}{R_{\rm S}^2}\right)^{-n} - 1\right]$	$\lambda(>0),\ n(>0),\ R_{\rm S}$
(iii) Tsujikawa	$R - \mu R_{\rm T}\tanh\left(\frac{R}{R_{\rm T}}\right)$	$\mu(>0),\ R_{\rm T}(>0)$
(iv) Exponential	$R - \beta R_{\rm E}\left(1 - e^{-R/R_{\rm E}}\right)$	$\beta,\ R_{\rm E}$

- $$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G_{eff}\,\rho_m\delta_m$$

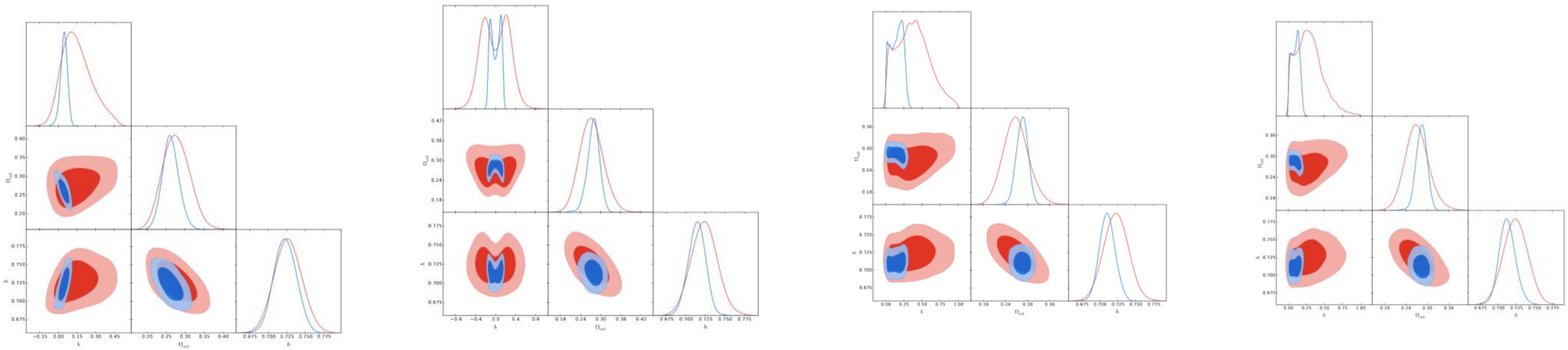
$$G_{eff} = \frac{G}{f'}\,\frac{1+4\frac{k^2}{a^2}\frac{f''}{f'}}{1+3\frac{k^2}{a^2}\frac{f''}{f'}}$$

$$\boxed{\frac{d\,\ln\delta_m}{d\,\ln a} = \Omega_m^{\gamma}(a)}$$





f(R) cosmology – Dark energy



Models	CC+ H_0				JLA + BAO + CC + H_0			
	AIC	ΔAIC	BIC	ΔBIC	AIC	ΔAIC	BIC	ΔBIC
ΛCDM Model	28.205	0	36.809	0	721.084	0	749.017	0
Hu-Sawicki Model	28.744	0.539	38.782	1.973	720.840	−0.244	753.428	4.411
Starobinsky Model	29.096	0.891	39.134	2.325	721.726	0.642	754.314	5.297
Tsujikawa Model	29.407	1.202	39.445	2.636	722.966	1.882	755.554	6.537
Exponential Model	29.310	1.105	39.347	2.538	722.548	1.464	755.136	6.119

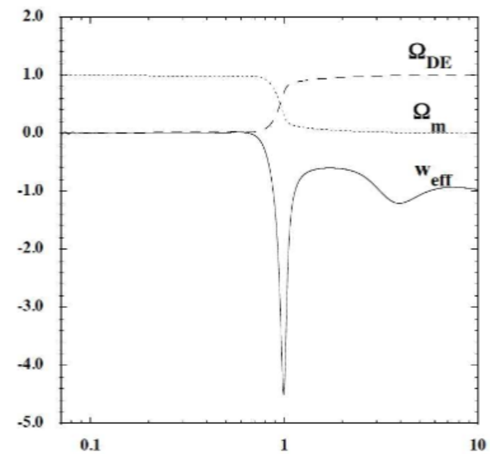
[Nunes, Pan, Saridakis, Abreu JCAP 1701]

f(G) Theories

- Gauss-Bonnet Invariant: $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$

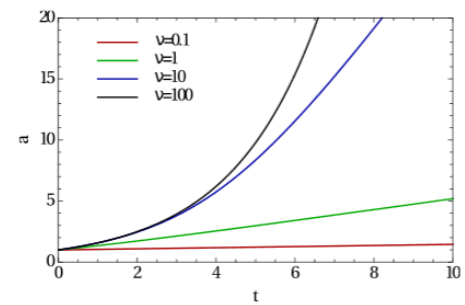
$$\boxed{S = \int \mathrm{d}^4x\sqrt{-g}\left[\tfrac{1}{2}R + f(\mathcal{G})\right] + S_m(g_{\mu\nu}, \Psi_m)}$$

$$\begin{aligned}
0 = {} & \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu} + \frac{1}{2}g^{\mu\nu}f(G) \\
& -2f'(G)RR^{\mu\nu} + 4f'(G)R^{\mu}{}_{\rho}R^{\nu\rho} \\
& -2f'(G)R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} - 4f'(G)R^{\mu\rho\sigma\nu}R_{\rho\sigma} \\
& +2\left(\nabla^{\mu}\nabla^{\nu}f'(G)\right)R - 2g^{\mu\nu}\left(\nabla^2 f'(G)\right)R \\
& -4\left(\nabla_{\rho}\nabla^{\mu}f'(G)\right)R^{\nu\rho} - 4\left(\nabla_{\rho}\nabla^{\nu}f'(G)\right)R^{\mu\rho} \\
& +4\left(\nabla^2 f'(G)\right)R^{\mu\nu} + 4g^{\mu\nu}\left(\nabla_{\rho}\nabla_{\sigma}f'(G)\right)R^{\rho\sigma} \\
& -4\left(\nabla_{\rho}\nabla_{\sigma}f'(G)\right)R^{\mu\rho\nu\sigma} .
\end{aligned}$$



$$\mathcal{G} = 24H^2(H^2 + \dot{H})\,.$$

$$3H^2 = \mathcal{G}f_{,\mathcal{G}} - f - 24H^3\dot{f}_{,\mathcal{G}} + \rho_m + \rho_{\mathrm{rad}}\,,$$



Cubic and f(P) Theories

- Cubic combination

$$P = \beta_1 R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} R_{\rho}{}^{\gamma}{}_{\sigma}{}^{\delta} R_{\gamma}{}^{\mu}{}_{\delta}{}^{\nu} + \beta_2 R^{\rho\sigma}_{\mu\nu} R^{\gamma\delta}_{\rho\sigma} R^{\mu\nu}_{\gamma\delta} + \beta_3 R^{\sigma\gamma} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho}{}_{\gamma} + \beta_4 R R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$
$$+\beta_5 R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + \beta_6 R^{\nu}_{\mu} R^{\rho}_{\nu} R^{\mu}_{\rho} + \beta_7 R_{\mu\nu} R^{\mu\nu} R + \beta_8 R^3,$$

$$\beta_7 = \frac{1}{12}(3\beta_1 - 24\beta_2 - 16\beta_3 - 48\beta_4 - 5\beta_5 - 9\beta_6),$$
$$\beta_8 = \frac{1}{72}(-6\beta_1 + 36\beta_2 + 22\beta_3 + 64\beta_4 + 3\beta_5 + 9\beta_6).$$

$$\beta_1 = \frac{14}{39}\beta_3 + 8\beta_4 - \frac{34}{39}\beta_5$$
$$\beta_2 = -\frac{11}{78}\beta_3 - \frac{1}{26}\beta_5$$
$$\beta_6 = \frac{2}{39}\beta_3 + 8\beta_4 - \frac{1}{13}\beta_5$$
$$\beta_7 = -\beta_3 - 8\beta_4 - \frac{1}{2}\beta_5,$$

$$S = \int \sqrt{-g}\, d^4x \left[\frac{1}{2\kappa}(R - 2\Lambda) + \alpha P\right],$$

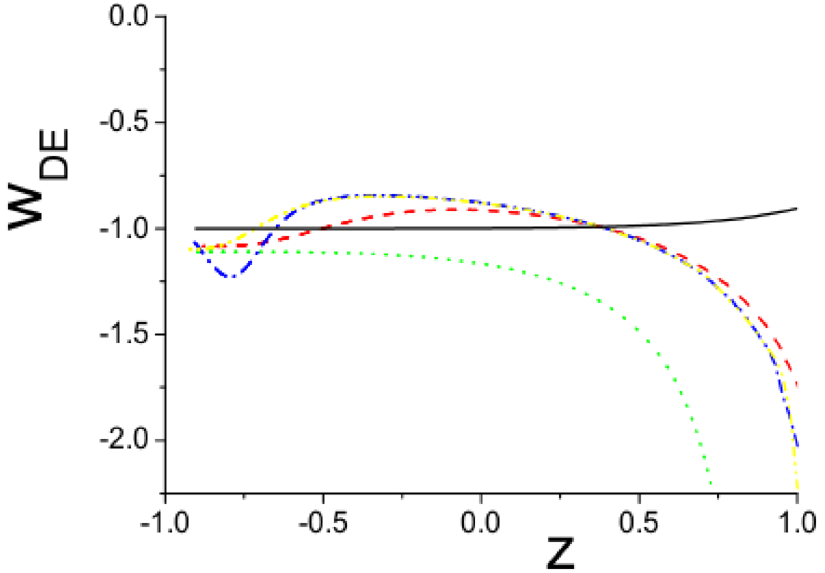
$$P = 6\tilde{\beta} H^4 (2H^2 + 3\dot{H}),$$

$$S = \int \sqrt{-g}\, d^4x \left[\frac{R}{2\kappa} + f(P)\right],$$

Cubic and f(P) Theories

$$3H^2 = \kappa\left(\rho_m + \rho_{cub}\right),$$
$$3H^2 + 2\dot{H} = -\kappa\left(p_m + p_{cub}\right),$$

$$\rho_{cub} \equiv 6\beta H^6 + \frac{\Lambda}{\kappa},$$
$$p_{cub} \equiv -6\beta H^4(H^2 + 2\dot{H}) - \frac{\Lambda}{\kappa},$$



Bi-scalar Theories

- Modified gravity propagating 2+2 dof's $\boxed{S=\int d^4x\sqrt{-g}\,f\left(R,(\nabla R)^2,\Diamond R\right)}$
- For $f\left(R,(\nabla R)^2,\Diamond R\right)=K\left(R,(\nabla R)^2\right)+Q\left(R,(\nabla R)^2\right)\Diamond R$

$$\boxed{\Rightarrow S=\int d^4x\sqrt{-g}\left[\frac{1}{2}\hat{R}-\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi-\frac{1}{\sqrt{6}}e^{-\sqrt{2/3}\chi}\hat{g}^{\mu\nu}Q\partial_\mu\chi\partial_\nu\phi+\frac{1}{4}e^{-2\sqrt{2/3}\chi}K+\frac{1}{2}e^{-\sqrt{2/3}\chi}Q\hat{\Diamond}\phi-\frac{1}{4}e^{-\sqrt{2/3}\chi}\phi\right]}$$

$$K=K(\phi,B),\ G=G(\phi,B),\ B=2e^{\sqrt{2/3}\chi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

Bi-scalar Theories

- Modified gravity propagating 2+2 dof's $$\boxed{S=\int d^4x\sqrt{-g}\,f\left(R,(\nabla R)^2,\Diamond R\right)}$$
- For $f\left(R,(\nabla R)^2,\Diamond R\right)=K\left(R,(\nabla R)^2\right)+Q\left(R,(\nabla R)^2\right)\Diamond R$

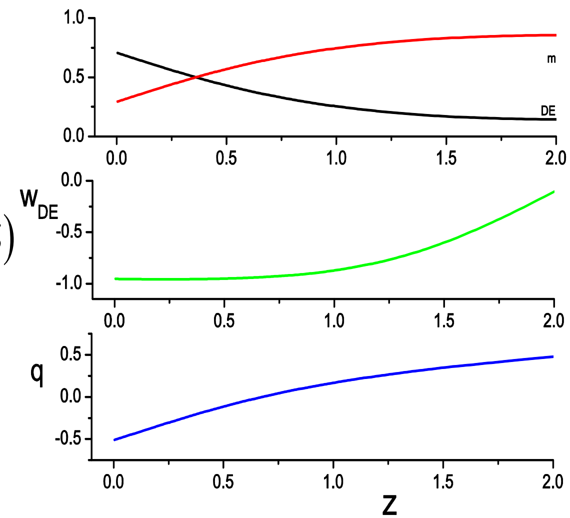
$$\boxed{\Rightarrow S=\int d^4x\sqrt{-g}\left[\frac{1}{2}\hat{R}-\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi-\frac{1}{\sqrt{6}}e^{-\sqrt{2/3}\chi}\hat{g}^{\mu\nu}Q\partial_\mu\chi\partial_\nu\phi+\frac{1}{4}e^{-2\sqrt{2/3}\chi}K+\frac{1}{2}e^{-\sqrt{2/3}\chi}Q\hat{\Diamond}\phi-\frac{1}{4}e^{-\sqrt{2/3}\chi}\phi\right]}$$

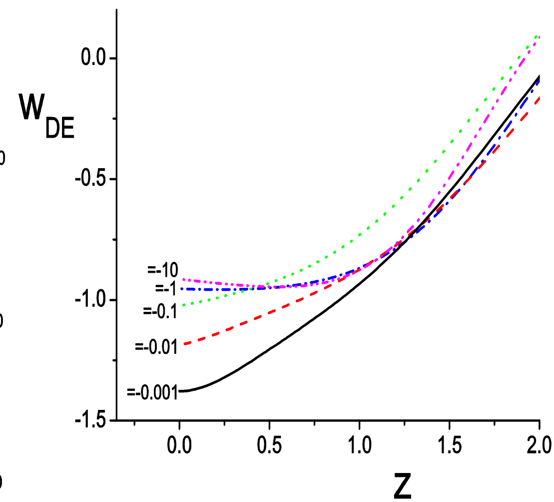
$$K=K(\phi,B),\; G=G(\phi,B),\; B=2e^{\sqrt{2/3}\chi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

- eg.: $K(\phi,B)=\frac{\phi}{2},\; G(\phi,B)=\xi B$

$$\rho_{DE}=\frac{1}{2}\dot{\chi}^2-\frac{1}{8}e^{-2\sqrt{2/3}\chi}\left(1-2e^{\sqrt{2/3}\chi}\right)\phi-\xi\dot{\phi}^3\left(\sqrt{6}\dot{\chi}-6H\right)$$

$$p_{DE}=\frac{1}{2}\dot{\chi}^2+\frac{1}{8}e^{-2\sqrt{2/3}\chi}\left(1-2e^{\sqrt{2/3}\chi}\right)\phi-\frac{1}{3}\xi\dot{\phi}^2\left(\sqrt{6}\dot{\phi}\dot{\chi}+6\ddot{\phi}\right)$$





MASSIVE GRAVITY

Introduction

- Massive Gravity, i.e adding mass to a spin-2 particle, goes back to 1939
- Motivation: i) Theoretical (we know the answer for scalars and vectors)

  ii) Cosmological (explain acceleration)
- Indeed it is the most reasonable modified gravity

  (not the simplest one, since you add 3 dof's)
- It is promising, but…

  [Hinterbichler, Rev.Mod.Phys.84]

Introduction

- 1939: Fierz and Pauli add a linear mass-term to GR $\propto m^2(h_{\mu\nu} - h^2)$
- 1970: van Dam, Veltman, Zakharov: When the linear theory couples to a source, the limit $\boxed{m \to 0}$ does not give GR

  (vDVZ discontinuity)

- 1972: Vainstein: The non-linearities become stronger and stronger as m decreases. They must be taken into account and they do cure vDVZ discontinuity
- 1972: Boulware, Deser: Nonlinearities bring a ghost!

Introduction

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- 1972: Boulware, Deser: Nonlinearities bring a ghost!
- 2010: de Rham, Gabadadze, Tolley: Adding higher-order graviton self-interaction systematically removes the BD ghost
- 2011 and on: The cosmology has severe problems.

Fierz-Pauli linear theory

- Put source $T^{\mu\nu}$ with coupling $\kappa\, h_{\mu\nu} T^{\mu\nu}$. Eoms':

$$\Diamond h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda_\nu - \partial_\lambda \partial_\nu h^\lambda_\mu + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Diamond h \boxed{- m^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h^2 \right)} = -\kappa\, T_{\mu\nu}$$

- Note: For $m = 0 \Rightarrow \partial^\mu T_{\mu\nu} = 0$ (conservation)

  For $m \neq 0$ no such condition (but we assume it, otherwise obvious discontinuity)

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  For $m \neq 0$ no such condition (but we assume it otherwise, obvious discontinuity)

- GR is NOT recovered in the massless limit (vDVZ discontinuity)

Massless gravity: 2 spin states

2 helicity states of a massless graviton

Massive gravity: 5 spin states

2 helicity states of a massless graviton

2 helicity states of a massless vector

1 single massive scalar

no 6th dof since the time components h_{00} appear as Lagr. multiplier

- The scalar (longitudinal graviton) maintains a coupling to T even in the massless limit
- I.e, the massless limit does not describe a massless graviton, but a massless graviton plus a coupled scalar
- The gauge symmetry of GR, that kills the extra dof appears ONLY for $m = 0$ and NOT for $m \to 0$

[van Dam, Veltman 1970], [Zakharov 1970]

Nonlinear theory and the BD ghost

- Nonlinearities become stronger as $m \to 0$, need to be taken into account.

$$S = \frac{1}{2\kappa^2}\underbrace{\int d^4x \quad [\sqrt{-g}R}_{\text{Full nonlinear EH action}} \boxed{\underbrace{-\sqrt{-g^{(0)}}\,\frac{1}{4}m^2 g^{(0)\mu\alpha}g^{(0)\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta}-h_{\mu\alpha}h_{\nu\beta}\right)\ ]}_{\text{Fierz-Pauli mass term}}}$$

Full nonlinear
EH action

Fierz-Pauli mass term

$g^{(0)}_{\mu\nu}$ the fixed metric on which the massive graviton propagates

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$g^{(0)}_{\mu\nu}$ the fixed metric on which the massive graviton propagates

- The nonlinearities re-bring the 6th dof (no Lagrange multiplier anymore)
- The Hamiltonian constraint analysis shows that it is a ghost!  [Boulware,Deser 1972]

- But this ghost cures the vDVZ discontinuity! (it provides a repulsive force that counteracts the attractive force of the longitudinal scalar mode) [Vainstein 1972]

- But it could still make sense, if quantum effects push the ghost above a cutoff Λ, and see the whole story as an effective theory [Arkani-Hamed, Georgi, Schwartz 2002]

dRGT nonlinear massive gravity

- The 6^{th} dof (ghost) survives since the lapse function N is not a Lagrange multiplier in the nonlinear case, as it was in the linear one.
- Idea: Specially design nonlinear terms, so that N becomes again a Lagrange multiplier

dRGT nonlinear massive gravity

- Finally:

$$\boxed{S_{MG} = M_p^2 \int d^4x \sqrt{-g}\left[\frac{R}{2} + m_g^2\left(L_2 + \alpha_3 L_3 + \alpha_4 L_4\right)\right]}$$

where

$$L_2 = \frac{1}{2}\left([K]^2 - [K^2]\right)$$

$$L_3 = \frac{1}{6}\left([K]^3 - 3[K][K^2] + 2[K^3]\right)$$

$$L_4 = \frac{1}{24}\left([K]^4 - 6[K]^2[K^2] + 3[K^2]^2 + 8[K][K^3] - 6[K^4]\right) \qquad [K] = tr\left(K_\mu^\nu\right)$$

$$K_\nu^\mu \equiv \delta_\nu^\mu - \sqrt{g^{\mu\sigma} \underset{\uparrow}{f_{ab}}(\phi)\partial_\nu \phi^a \partial_\sigma \phi^b}$$

fiducial metric　　　Stückelberg fields

[de Rham, Gabadadze, PRD 82],
[de Rham, Gabadadze, Tolley PRL 106]

- Free of BD ghost! Free of vDVZ discontinuity!
- Vainstein mechanism: extra dof's are suppressed at small scales due to non-linearities

Cosmological applications

- Simplest Example: Physical metric: flat FRW: $ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$

  Fiducial metric: Minkowski: $f_{ab} = \eta_{ab}$

  Stückelberg scalars: $\phi^0 = b(t), \ \phi^i = x^i$

Variation wrt ϕ : $m^2\partial_0(a^3 - a^2) = 0 \Rightarrow \dot{a} = 0$ NO nontrivial solution (same for closed)

[dRGT et al, PRD 84]

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[dRGT et al, PRD 84]

- Next: Physical metric open FRW: $ds^2 = -N^2dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2 - \frac{|K|(xdx + ydy + zdz)^2}{1 + |K|(x^2 + y^2 + z^2)}\right]$

  Fiducial metric: Minkowski: $f_{ab} = \eta_{ab}$

  Stückelberg scalars: $\phi^0 = b(t)\sqrt{1 + |K|(x^2 + y^2 + z^2)}, \ \phi^i = \sqrt{|K|}b(t)x^i$

Variation wrt ϕ gives a constraint for b(t): $\frac{b(t)}{a(t)} = \frac{X_\pm}{\sqrt{|K|}} = const., \ X_\pm = \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4}$

- Friedmann equations:

$$3H^2 - 3\frac{|K|}{a^2} = \rho_m + m_g^2 c_\pm$$

$$-2\dot{H} - 2\frac{|K|}{a^2} = \rho_m + p_m$$

We get an Effective Cosmological Constant: $\boxed{\Lambda_\pm = m_g^2 c_\pm(\alpha_3, \alpha_4)}$

Self-acceleration for $c_\pm(\alpha_3, \alpha_4) > 0$

Perturbations

- Unfortunately, there is ALWAYS a ghost instability (it's frequency tends to vanish at low scales so it always remain in the low-energy effective theory)

- This instability is related to the FRW structure of the physical metric, and in particular from the high symmetries (isotropy).

[Gumrukcuoglu, Lin, Mukohyama, JCAP1203]

Perturbations

- Unfortunately, there is ALWAYS a ghost instability (it's frequency tends to vanish at low scales so it always remain in the low-energy effective theory)

- This instability is related to the FRW structure of the physical metric, and in particular from the high symmetries (isotropy).

- In order to construct a healthy model we must insert anisotropies:

Physical metric: axisymmetric Bianchi I: $ds^2 = -N^2 dt^2 + a(t)^2\left(e^{4\sigma(t)}dx^2 + e^{-2\sigma(t)}dy^2 + e^{-2\sigma(t)}dz^2\right)$

Fiducial metric: FRW: as before

Stückelberg scalars: as before

$$\boxed{\Longrightarrow \rho_{MG}(t) = \cdots}$$

[Gumrukcuoglu, Lin, Mukohyama, PLB717]

- The only healthy model. Disadvantage: There is NO isotropic limit!

Extension 1: Varying mass massive gravity

- Need to find extensions of nonlinear massive gravity where FRW solutions are stable.

$$\boxed{S_{MG} = M_p^2 \int d^4x \sqrt{-g}\left[\frac{R}{2} + V(\psi)\left(L_2 + \alpha_3 L_3 + \alpha_4 L_4\right) - \frac{1}{2}\partial_\mu \psi \partial^\mu \psi - W(\psi)\right]}$$

[Huang, Piao, Zhou PRD86]

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  Fiducial metric: Minkowski: $f_{ab} = \eta_{ab}$

  Stückelberg scalars: $\phi^0 = b(t), \ \phi^i = a_{ref} x^i$

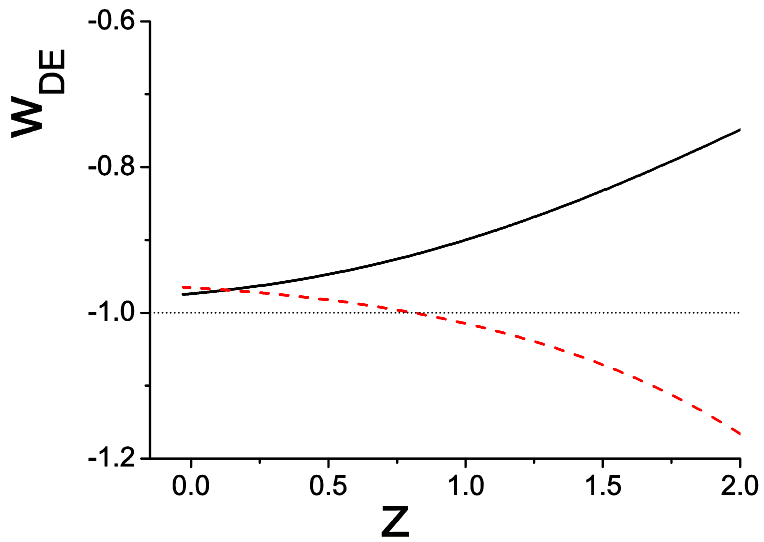
$$\Rightarrow 3M_p^2 H^2 = \rho_m + \rho_{MG}$$
$$-2M_p^2 \dot{H} = \rho_m + p_m + \rho_{MG} + p_{MG}$$

$$\rho_{MG} = \frac{1}{2}\dot{\psi}^2 + W(\psi) + V(\psi)\left(\frac{a_{ref}}{a} - 1\right)\left[f_3(a) + f_1(a)\right]$$

$$p_{MG} = \frac{1}{2}\dot{\psi}^2 - W(\psi) - V(\psi)\left[f_4(a) + \dot{b} f_1(a)\right]$$

$$w_{DE} = \frac{p_{MG}}{\rho_{MG}}$$

$$\rho_{MG} + p_{MG} = \dot{\psi}^2 - V(\psi)\left(\dot{b} - \frac{a_{ref}}{a}\right) f_1(a)$$



[Saridakis CQG 30]

Extension 2: Quasi-dilaton massive gravity

$$\boxed{S_{MG} = M_p^2 \int d^4x \sqrt{-g}\left[\frac{R}{2} + m_g^2\left(L_2 + \alpha_3 L_3 + \alpha_4 L_4\right) - \frac{\omega}{2M_p^2} g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right]}$$

where

$$L_2 = \frac{1}{2}\left([K]^2 - [K^2]\right)$$

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$$K^\mu_\nu \equiv \delta^\mu_\nu - e^{\sigma/M_p}\sqrt{g^{\mu\sigma}\eta_{ab}(\phi)\partial_\nu\phi^a\partial_\sigma\phi^b}$$

quasi-dilaton fiducial metric Stückelberg fields

[D'Amico, Gabadadze, Hui, Pirtskhalava PRD 87]

Extension 2: Quasi-dilaton massive gravity

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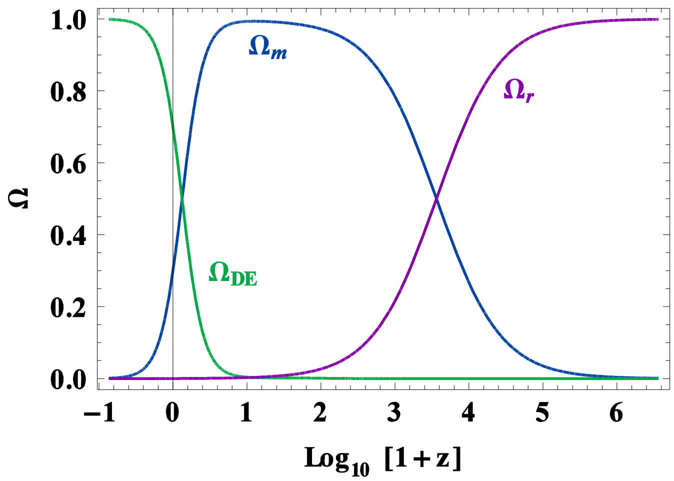
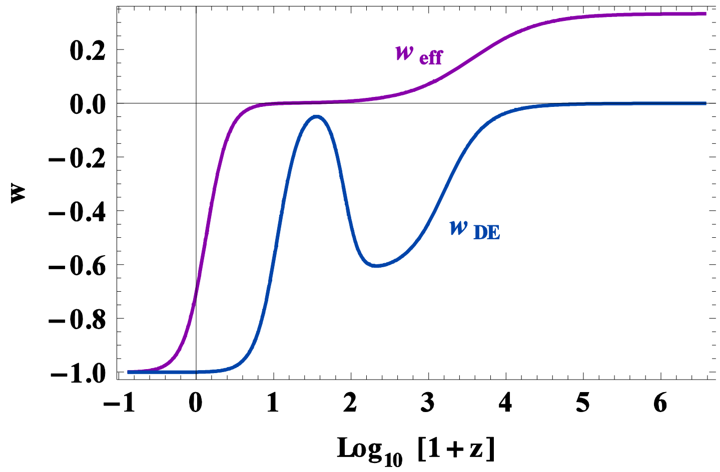
  Fiducial metric: Minkowski: $f_{ab} = \eta_{ab}$
  Stückelberg scalars: $\phi^0 = b(t), \;\; \phi^i = x^i$

$$\Rightarrow 3M_p^2 H^2 = \rho_m + \rho_r + \rho_{DE}$$

$$\rho_{DE} = \frac{\omega}{2}\dot{\psi}^2 - 3M_p^2 m_g^2\left[(2+\alpha_3+\alpha_4) - \left(3+\frac{9}{4}\alpha_3+3\alpha_4\right)\frac{e^{\frac{\sigma}{M_p}}}{a} + \left(1+\frac{3}{2}\alpha_3+3\alpha_4\right)\frac{e^{\frac{2\sigma}{M_p}}}{a^2} - \frac{1}{4}(\alpha_3+4\alpha_4)\frac{e^{\frac{3\sigma}{M_p}}}{a^3}\right]$$

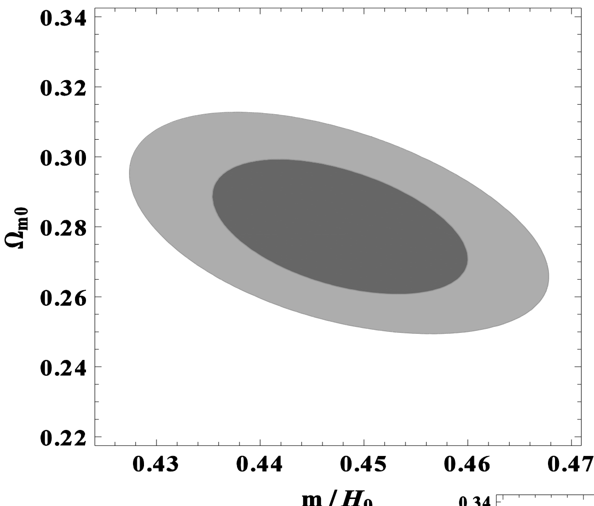


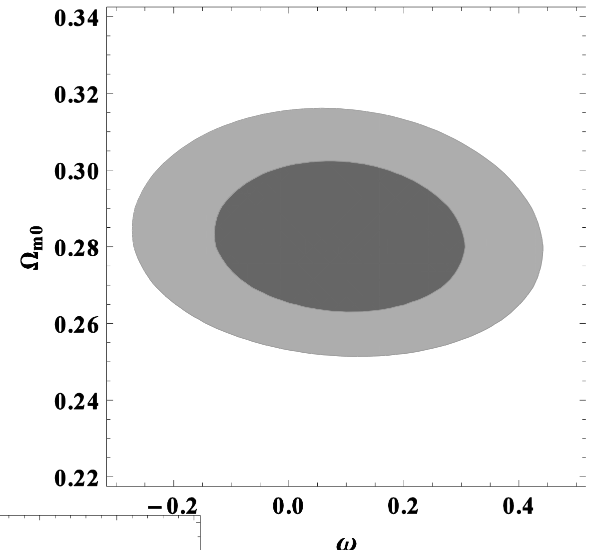


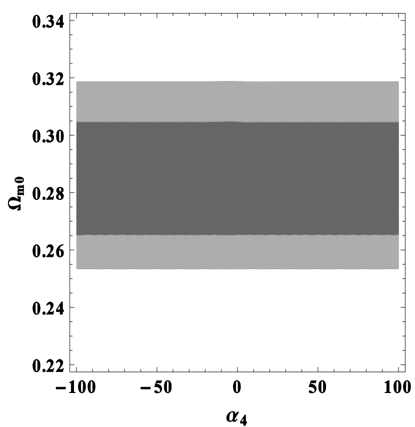
[Gannouji, Hossain, Sami, Saridakis PRD 87]

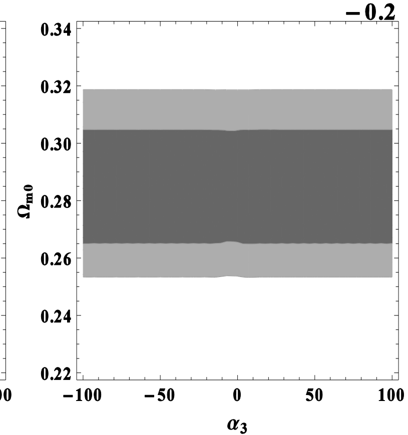
Observational constraints on quasi-dilaton massive gravity

- Use observational data (SNIa, BAO, CMB) to constrain the parameters of the theory. We fit $\Omega_{M0}, \Omega_{DE0}, m_g, \omega, \alpha_3, \alpha_4$









[Gannouji, Hossain, Sami, Saridakis PRD 87]

Extension 3: F(R) nonlinear massive gravity

$$S = M_p^2 \int d^4x \sqrt{-g}\left[\frac{F(R)}{2} + m_g^2\left(L_2 + \alpha_3 L_3 + \alpha_4 L_4\right)\right]$$

UV modification   IR modification

where

$$L_2 = \frac{1}{2}\left([K]^2 - [K^2]\right)$$

$$L_3 = \frac{1}{6}\left([K]^3 - 3[K][K^2] + 2[K^3]\right)$$

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$$K_\nu^\mu \equiv \delta_\nu^\mu - \sqrt{g^{\mu\sigma} f_{ab}(\phi)\partial_\nu \phi^a \partial_\sigma \phi^b}$$

[Cai, Duplessis, Saridakis PRD 90a]   [Cai, Saridakis PRD 90b]

Extension 3: F(R) nonlinear massive gravity

- Einstein frame: $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with $\Omega^2 = F_{,R} = \exp\left(\sqrt{\frac{2}{3}}\frac{\varphi}{M_p}\right)$

$$\boxed{S = \int d^4x\sqrt{-g}\left[M_p^2\frac{\tilde{R}}{2} + M_p^2 m_g^2\left(\tilde{L}_2 + \alpha_3\tilde{L}_3 + \alpha_4\tilde{L}_4\right) - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - U(\varphi)\right]}$$

with $U(\varphi) = M_p^2 \frac{RF_{,R} - F}{2F_{,R}^2}$

- Hamiltonian constraint analysis: the BD ghost is removed similar to usual nonlinear massive gravity

- Much more general than other massive gravity extensions.

[Cai, Duplessis, Saridakis PRD 90a] [Cai, Saridakis PRD 90b]

Cosmology of F(R) nonlinear massive gravity

- Physical metric: open FRW: $ds^2 = -N^2 dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2 - \frac{|K|(xdx + ydy + zdz)^2}{1+|K|(x^2+y^2+z^2)}\right]$
  Fiducial metric: Minkowski: $f_{ab} = \eta_{ab}$
  Stückelberg scalars: $\phi^0 = b(t)\sqrt{1+|K|(x^2+y^2+z^2)},\ \phi^i = \sqrt{|K|}b(t)x^i$

- Variation wrt b provides the constraint equation with solution: $\frac{b(t)}{a(t)} = const.$

$$3M_p^2\left(H^2 - \frac{|K|}{a^2}\right) = \rho_m + \rho_{MG} + \rho_{F_R}$$

$$\left.\begin{aligned}\rho_{MG} &= m_g^2 c_\pm \\ \rho_{F_{,R}} &= M_p^2\left[\frac{RF_{,R} - F}{2} - 3H\dot{R}F_{,RR}\right]\end{aligned}\right\}$$

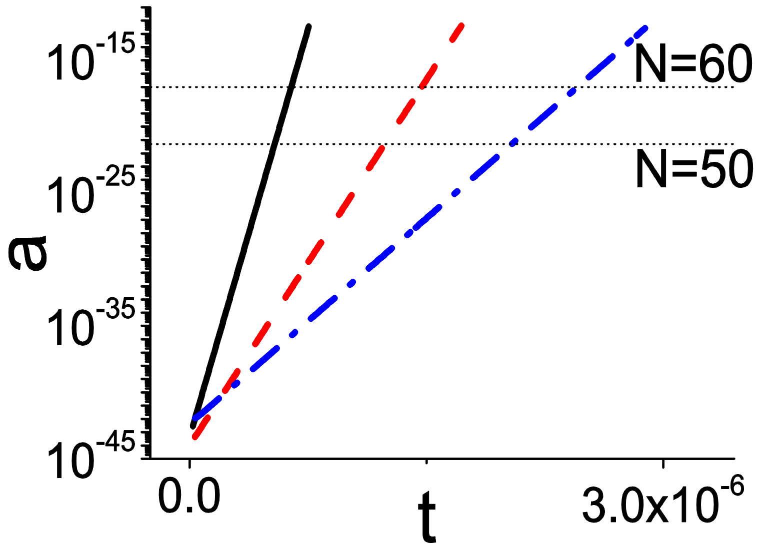
$$\boxed{\rho_{DE} \equiv \rho_{MG} + \rho_{F_R}}$$

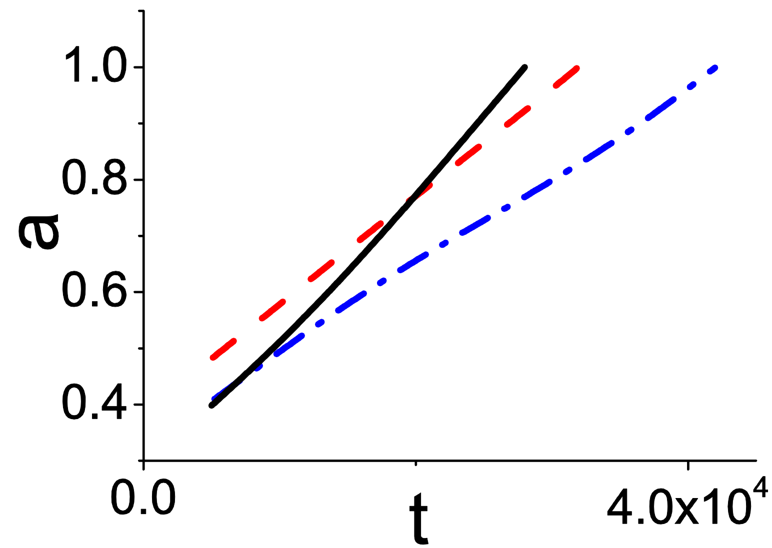
- Both IR and UV gravity modifications play a role in universe evolution.
- Huge capabilities.

[Cai, Duplessis, Saridakis PRD 90a] [Cai, Saridakis PRD 90b]

Cosmology of F(R) nonlinear massive gravity

- 1) $F(R) = R + \frac{\xi}{M_p^2} R^2$



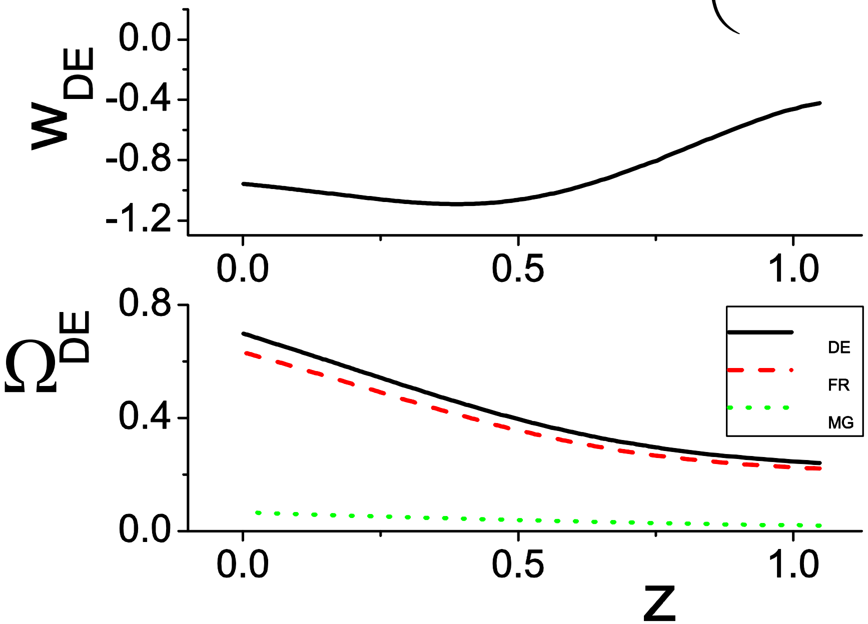


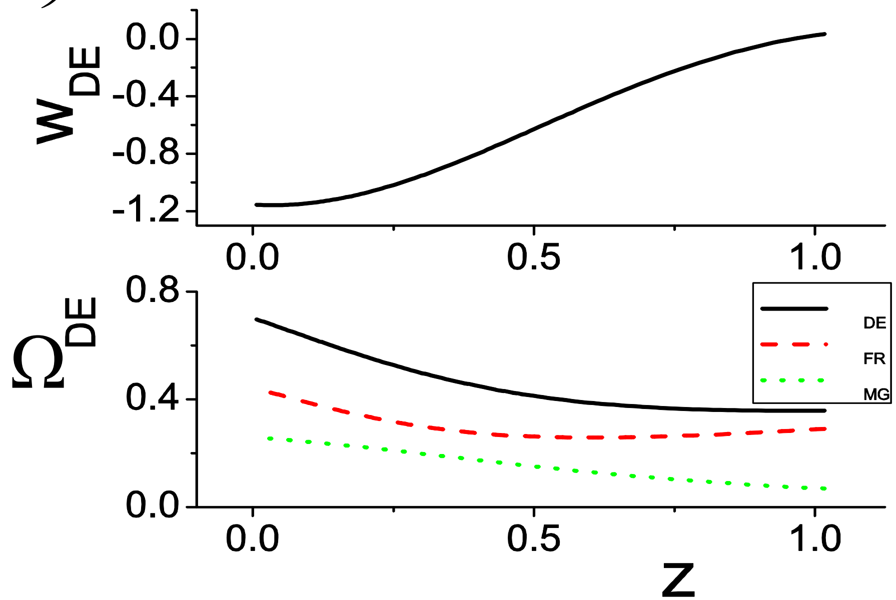
- Early times: F(R) sector drives inflation
- Late times: MG sector drives late-time acceleration

[Cai, Duplessis, Saridakis PRD 90a]

Cosmology of F(R) nonlinear massive gravity

- 2) $F(R) = R - \beta R_S \left(1 - e^{-\frac{R}{R_S}}\right)$



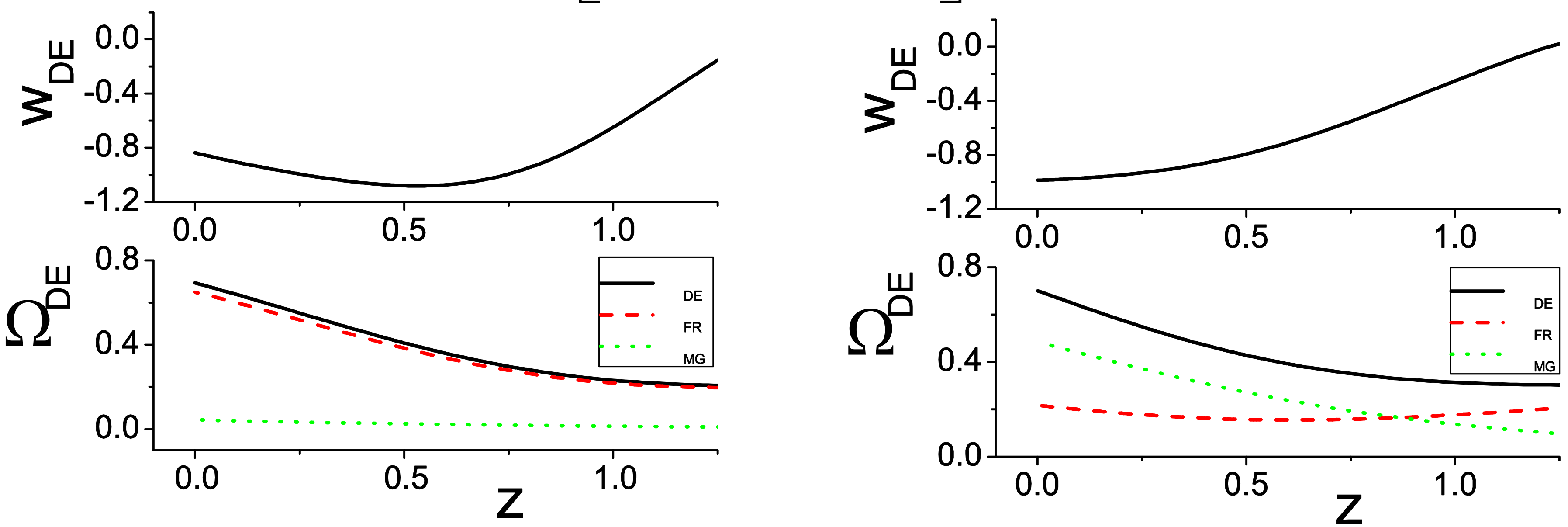


- Both F(R) sector and MG sector constitute Dark Energy $\rho_{DE} \equiv \rho_{MG} + \rho_{F_R}$
- w_{DE} can lie in the phantom regime.

[Cai, Saridakis PRD 90b]

Cosmology of F(R) nonlinear massive gravity

- 3) $F(R) = R - \lambda R_C \left[1 - \left(1 + \frac{R^2}{R_C^2} \right)^{-n} \right]$



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- w_{DE} can lie in the phantom regime.

[Cai, Saridakis PRD 90b]

Cosmological Perturbations

- $$\boxed{S=\int d^4x\sqrt{-\tilde{g}}\left[M_p^2\frac{\tilde{R}}{2}+M_p^2 m_g^2\left(\tilde{L}_2+\alpha_3\tilde{L}_3+\alpha_4\tilde{L}_4\right)-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-U(\varphi)\right]}$$

$$\delta\tilde{g}_{00}=-2N^2\phi,\ \ \delta\tilde{g}_{0i}=Na\partial_i B,\ \ \delta\tilde{g}_{ij}=a^2\left[2\tilde{\gamma}^K_{ij}\psi+\left(\nabla_i\nabla_j-\frac{1}{3}\tilde{\gamma}^K_{ij}\nabla_k\nabla^k\right)\right]E,\ \ \delta\varphi$$

$$\Rightarrow\cdots\ \ \cdots$$

- Integrate out non-dynamical dof's $\phi,\ B,\ E$
- Since ϕ is non-dynamical at the linear level on the self-accelerating solution, we introduce the Bardeen potential ψ_B and Mukkanov-Sasaki variable $\boxed{Q\equiv\delta\varphi+\frac{\dot{\varphi}\psi_B}{H}}$

$$\Rightarrow\underbrace{\ddot{Q}_k+3H\dot{Q}_k+\left[\frac{k^2}{a^2}+U_{,\varphi\varphi}-\frac{1}{M_p^2a^3}\left(\frac{a^3}{H}\dot{\varphi}^2\right)^{\bullet}\right]Q_k}_{\text{GR + scalar}}=\underbrace{\frac{2m_g^2\tilde{Y}_Q}{3\Omega^4}Q_k}_{\text{MG contribution}}-2\frac{k^2}{a^2H^2}\left(\ddot{\varphi}-\frac{\dot{H}\dot{\varphi}}{H}\right)\psi_B$$

- $\tilde{Y}_Q(\alpha_3,\alpha_4)<0\Rightarrow$ Stability!

[Cai, Duplessis, Saridakis PRD 90a] [Cai, Saridakis PRD 90b]

Status of massive gravity

- i) Massive gravity is a reasonable modification to describe acceleration.
- ii) The simplest linear model has the vDVZ discontinuity.
- iii) Non-linearities cure it but bring the BD ghost.
- iv) New nonlinear MG uses suitable graviton self-interactions in order to be free of BD ghosts and vDVZ discontinuity.
- v) But simple FRW cosmology is impossible (cosmological instabilities).
- vi) One should go to anisotropic geometry.
- vii) Or other extensions: Varying mass massive gravity, quasi-dilaton massive gravity.
- viii) F(R) nonlinear massive gravity is the most promising. It is free of BD ghost and vDVZ discontinuity. It exhibits good and rich cosmology, free of instabilities!

Re-parametrization of our ignorance? (instead to explain why Λ is small, we have to explain why m_g is small).

An interesting possibility: Horava-Lifshitz gravity

An interesting possibility: Horava-Lifshitz gravity

- **Horava-Lifshitz** gravity: power-counting renormalizable, UV complete
- IR fixed point: General Relativity
- Good UV behavior: Anisotropic, Lifshitz scaling between time and space

[Horava, PRD 79]

- Theoretical and conceptual problems (instabilities etc)?
- Open subject.

Horava-Lifshitz gravity

$$ds^2 = -N^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

$$\boxed{t \to l^3 t,\ x^i \to l x^i}$$

$$S_g = \int dt d^3x \sqrt{g} N \{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2)$$

$$+ \frac{\kappa^2}{2w^4} C_{ij} C^{ij} - \frac{\kappa^2 \mu \varepsilon^{ijk}}{2w^2 \sqrt{g}} R_{il} \nabla_j R_k^l + \frac{\kappa^2 \mu^2}{8} R_{il} R^{ij}$$

$$+ \frac{\kappa^2 \mu^2}{8(1-3\lambda)} \left[\frac{1-4\lambda}{4} R^2 + \Lambda R - 3\Lambda^2 \right] \}$$

(detailed-balanced)

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$$

(extrinsic curvature)

$$C^{ij} = \frac{\varepsilon^{ijk}}{\sqrt{g}} \nabla_k \left(R_i^j - \frac{1}{4} R \delta_i^j \right)$$

(Cotton tensor)

Horava-Lifshitz cosmology

- Cosmological framework:

$$N=1\ ,\ \ g_{ij}=a^2(t)\gamma_{ij}\ ,\ \ N^i=0 \qquad \text{(projectability)}$$

$$\gamma_{ij}dx^i dx^j=\frac{dr^2}{1-kr^2}+r^2 d\Omega_2^2$$

- Friedmann Equations (under detailed balance):

$$\boxed{H^2=\frac{\kappa^2}{6(3\lambda-1)}\left(\rho_M+\frac{3\kappa^2\mu^2k^2}{8(3\lambda-1)a^4}+\frac{3\kappa^2\mu^2\Lambda^2}{8(3\lambda-1)}\right)-\frac{\kappa^4\mu^2\Lambda}{8(3\lambda-1)^2}\frac{k}{a^2}}$$

$$\boxed{\dot{H}+\frac{3}{2}H^2=-\frac{\kappa^2}{4(3\lambda-1)}\left(w_M\rho_M+\frac{\kappa^2\mu^2k^2}{8(3\lambda-1)a^4}-\frac{3\kappa^2\mu^2\Lambda^2}{8(3\lambda-1)}\right)-\frac{\kappa^4\mu^2\Lambda}{16(3\lambda-1)^2}\frac{k}{a^2}}$$

- Effective dark energy:

$$\boxed{\rho_{DE}=\frac{3\kappa^2\mu^2k^2}{8(3\lambda-1)a^4}+\frac{3\kappa^2\mu^2\Lambda^2}{8(3\lambda-1)}}$$

$$\boxed{p_{DE}=\frac{\kappa^2\mu^2k^2}{8(3\lambda-1)a^4}-\frac{3\kappa^2\mu^2\Lambda^2}{8(3\lambda-1)}}$$

$$\Rightarrow \dot{\rho}_{DE}+3H(\rho_{DE}+p_{DE})=0$$

$$G\equiv\frac{\kappa^2}{16\pi(3\lambda-1)}$$

$$\frac{\kappa^4\mu^2\Lambda}{8(3\lambda-1)^2}\equiv 1$$

[Leon, Saridakis, JCAP 0911]

QUANTUM GRAVITY?

Maxwell Theory

Field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Maxwell action (in vacuum):

$$S = -\frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu}$$

Gauge invariance: $A_\mu \to A_\mu + \partial_\mu \Lambda$

Canonical Quantization of Electromagnetism

Fix gauge: Lorenz gauge $\partial^\mu A_\mu = 0$

Canonical variables: A_i, $\pi^i = -F^{0i}$

Equal-time commutation:

$$[A_i(\vec{x}, t), \pi^j(\vec{y}, t)] = i\delta_i^j \delta^3(\vec{x} - \vec{y})$$

Mode Expansion and Fock Space

Plane-wave expansion:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \sum_{\lambda=1}^{2} \left[\epsilon_\mu^{(\lambda)}(k) a_{\vec{k},\lambda} e^{-ik\cdot x} + \text{h.c.} \right]$$

Photon Fock space from creation/annihilation operators $a^\dagger_{\vec{k},\lambda}$

Path Integral Quantization

Partition function:

$$Z = \int \mathcal{D}A_\mu \, e^{iS[A]}$$

Gauge fixing (Faddeev–Popov):

$$Z = \int \mathcal{D}A_\mu \delta(\partial^\mu A_\mu) \det\left(\partial^2\right) e^{iS[A]}$$

Perturbation theory around free field

Quantum Electrodynamics

Lagrangian:

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

Gauge invariance under $U(1)$ symmetry

Renormalizable: divergences controlled by finite number of counterterms

What is Renormalization?

Loop diagrams in QFT lead to UV divergences:

$$\int d^4 k \to \infty$$

Renormalization introduces counterterms to absorb infinities:

$$\mathcal{L}_{\text{bare}} = \mathcal{L}_{\text{ren}} + \delta\mathcal{L}$$

Redefine mass, charge, and fields:

$$e_0 = Z_e e, \quad m_0 = Z_m m$$

Why Renormalization Works in QED

Gauge invariance tightly constrains divergences

Only a few divergent structures $\Rightarrow$ finite number of counterterms

QED is renormalizable in 4D: predictive theory at all scales

Beta function governs running coupling:

$$\mu \frac{de}{d\mu} = \beta(e)$$

Why QED Works

Free theory: linear equations, stable vacuum

Interactions weak (fine-structure constant $\alpha \approx 1/137$)

Perturbative expansion converges well

Renormalizability ensures predictivity

General Relativity: A Geometric Theory

Einstein–Hilbert action:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R$$

Field equations (second order, non-linear):

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Diffeomorphism invariance: coordinate-free structure

Attempting Canonical Quantization

ADM variables: h_{ij}, π^{ij} on spatial slice

Constraints:

$$\mathcal{H}_0 = 0, \quad \mathcal{H}_i = 0$$

Wheeler–DeWitt equation:

$$\hat{\mathcal{H}}\Psi[h_{ij}] = 0$$

Problems: No clear time evolution, operator ordering ambiguity

Path Integral Approach to Gravity

Formal analogy:

$$Z = \int \mathcal{D}g_{\mu\nu}\, e^{iS[g]}$$

Problem: $S[g]$ is not bounded below $\Rightarrow$ non-convergent integral

Gauge fixing is far more complex (diffeomorphism group)

Gravity is Non-Renormalizable

Graviton propagator from expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

Interaction terms grow with energy

Perturbative loop corrections diverge badly

Requires infinite counterterms $\Rightarrow$ no predictivity

Why Renormalization Fails in Gravity

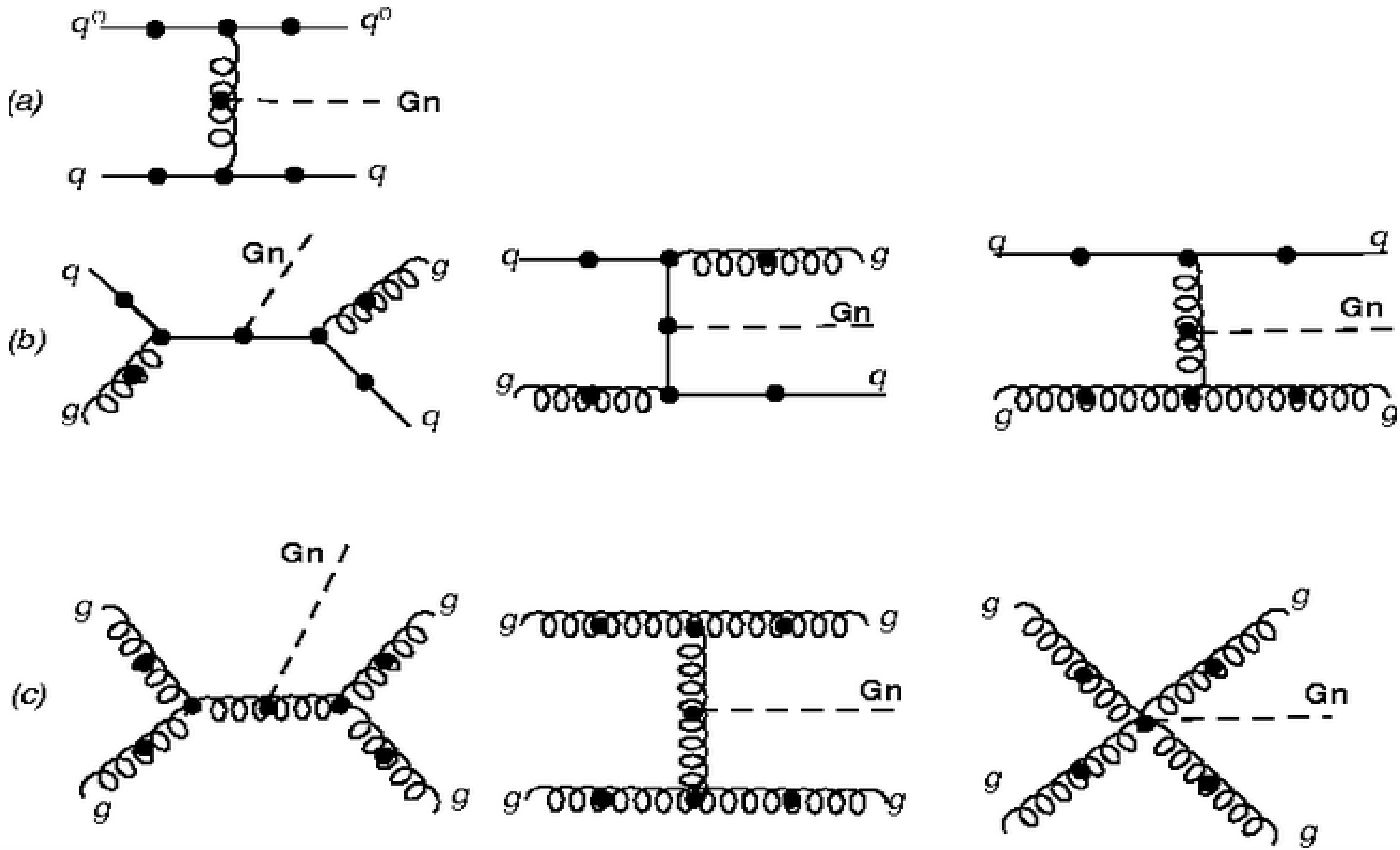
Newton's constant G has dimension $[G] = (\text{mass})^{-2}$

Higher loops generate new divergences (e.g. R^2, $R_{\mu\nu}R^{\mu\nu}$, etc.)

Cannot absorb into finite set of parameters

Theory is only valid as low-energy effective field theory

Can General Relativity be quantized?



COSMOLOGICAL CONSTANT PROBLEM

$$E_n \sim (n + 1/2) h\omega(k)$$

$$\rho_\Lambda(th) \sim M_p^4$$

$$\rho_\Lambda^0 \sim 10^{-120} \rho_\Lambda^{th}$$

Why Quantizing Gravity Fails (Perturbatively)

Gravity is inherently non-linear and self-couples

Coupling constant G has negative mass dimension

Quantum fluctuations of spacetime have no fixed background

Problem of time: no absolute time in GR

EM: linear, gauge theory, perturbatively renormalizable

GR: geometric, diffeomorphism invariant, non-renormalizable

Canonical and path integral methods fail in gravity

Non-perturbative approaches: LQG, string theory, asymptotic safety

Why Quantize Gravity?

Incompatibility of General Relativity (GR) and Quantum Field Theory (QFT)

Black hole entropy and evaporation demand a quantum description

GR breaks down at singularities: need UV completion

The Hamiltonian Formulation of GR

Einstein-Hilbert action:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R$$

ADM decomposition:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Canonical momentum:

$$\pi^{ij} = \frac{\delta \mathcal{L}}{\delta \dot{h}_{ij}}$$

Constraints and Wheeler-DeWitt Equation

Hamiltonian constraint:

$$\mathcal{H} = \frac{1}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \sqrt{h}R = 0$$

Momentum constraint:

$$\mathcal{H}_i = -2\nabla_j \pi^j_i = 0$$

Quantization:

$$\hat{\mathcal{H}}\Psi[h_{ij}] = 0$$

Motivation for LQG

Non-perturbative, background-independent quantization of GR

Reformulates GR as SU(2) gauge theory using Ashtekar variables

Avoids UV divergences of perturbative quantum gravity

Ashtekar Variables

Canonical pair:

$$A_a^i = \Gamma_a^i + \gamma K_a^i$$
$$E_i^a = \text{densitized triad}, \quad \det(E_i^a) = h$$

Poisson bracket:

$$\{A_a^i(x), E_j^b(y)\} = 8\pi G\gamma \delta_a^b \delta_j^i \delta^3(x-y)$$

Spin Networks and Kinematical Hilbert Space

Quantum states are functions of holonomies: $\Psi[A] = \psi(h_\gamma[A])$

Spin networks: eigenstates of geometric operators

Nodes: intertwiner labels; edges: SU(2) representations

Geometric Operators in LQG

Area operator:

$$\hat{A}(S) = 8\pi\gamma\ell_P^2 \sum_i \sqrt{j_i(j_i+1)}$$

Volume operator:

$$\hat{V}(R) = \int_R \sqrt{\left|\frac{1}{3!}\epsilon_{abc}\epsilon^{ijk} E_i^a E_j^b E_k^c\right|} d^3x$$

Semi-Classical Limit of LQG

Coherent states: peaked around classical geometry

Use of expectation values:

$$\langle \Psi | \hat{O} | \Psi \rangle \approx O_{\text{classical}}$$

Recover GR in the large spin limit $j \gg 1$

Basic Ideas of String Theory

Strings replace particles:

$$X^{\mu}(\sigma, \tau) : \quad \text{worldsheet} \to \text{spacetime}$$

Vibrational modes $\Rightarrow$ mass spectrum

Graviton arises as a closed string excitation

String Actions

Polyakov action:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$$

Conformal invariance $\Rightarrow$ conditions on $g_{\mu\nu}$

Quantization and Low Energy Limit

Quantize X^μ: oscillator modes

Low energy effective action:

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}H^2 + \dots\right)$$

From Strings to GR

String theory defines scattering amplitudes $\mathcal{A}$

Low-energy limit gives effective action:

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_P^2 R + \ldots \right]$$

GR emerges as leading term in derivative expansion

Supersymmetry and Supergravity

Local supersymmetry $\Rightarrow$ supergravity

String theory low-energy limit matches 10D SUGRA

Compactification $\Rightarrow$ 4D N=1 SUGRA + matter

Compactification and Extra Dimensions

Extra dimensions compactified on Calabi-Yau 3-folds

Metric ansatz:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{mn}(y) dy^m dy^n$$

Moduli determine low-energy couplings

Branes and Gauge/Gravity Duality

D-branes support open strings: gauge fields

Closed strings propagate in bulk: gravity

AdS/CFT:

$$\text{String theory on } AdS_5 \times S^5 \Leftrightarrow \mathcal{N} = 4 \; SU(N) \text{ SYM}$$

Summary and Challenges

Canonical QG: conceptual problems (e.g., time, observables)

LQG: robust kinematics, but dynamics and low-energy limit challenging

Strings: UV complete, unifying, but needs compactification

Goal: recover classical GR in effective limit

TORSIONAL GRAVITY

“Those that do not know geometry are not allowed to enter”.
Front Door of Plato’s Academy

Descriptions of Gravity

- Einstein 1916: **General Relativity**:
  energy-momentum source of spacetime Curvature
  Levi-Civita connection: Zero Torsion

- Einstein 1928: **Teleparallel Equivalent of GR:**
  Weitzenbock connection: Zero Curvature

[Cai, Capozziello, De Laurentis, Saridakis, Rept.Prog.Phys. 79]

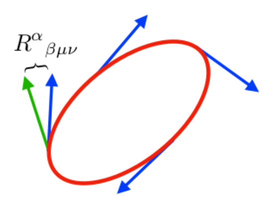
$$\left\{ {}^{\ \alpha}_{\mu\nu} \right\} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda} \right). \tag{1.3}$$

The corresponding covariant derivative will be denoted by $\mathcal{D}$ so that we will have $\mathcal{D}_\alpha g_{\mu\nu} = 0$. A general connection $\Gamma^\alpha{}_{\mu\nu}$ then admits the following convenient decomposition:

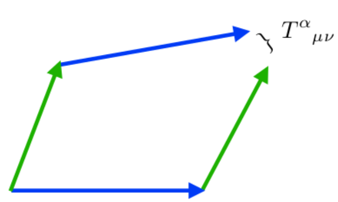
$$\Gamma^\alpha{}_{\mu\nu} = \left\{ {}^{\ \alpha}_{\mu\nu} \right\} + K^\alpha{}_{\mu\nu} + L^\alpha{}_{\mu\nu} \tag{1.4}$$

with

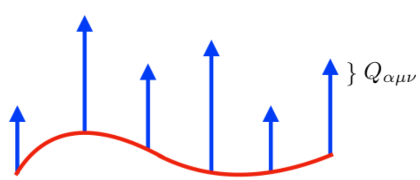
$$K^\alpha{}_{\mu\nu} = \frac{1}{2} T^\alpha{}_{\mu\nu} + T_{(\mu}{}^\alpha{}_{\nu)}\,, \quad L^\alpha{}_{\mu\nu} = \frac{1}{2} Q^\alpha{}_{\mu\nu} - Q_{(\mu}{}^\alpha{}_{\nu)} \tag{1.5}$$



The rotation of a vector transported along a closed curve is given by the curvature: General Relativity.



The non-closure of parallelograms formed when two vectors are transported along each other is given by the torsion: Teleparallel Equivalent of General Relativity.



The variation of the length of a vector as it is transported is given by the non-metricity: Symmetric Teleparallel Equivalent of General Relativity.

Metric-Affine Modified Gravity

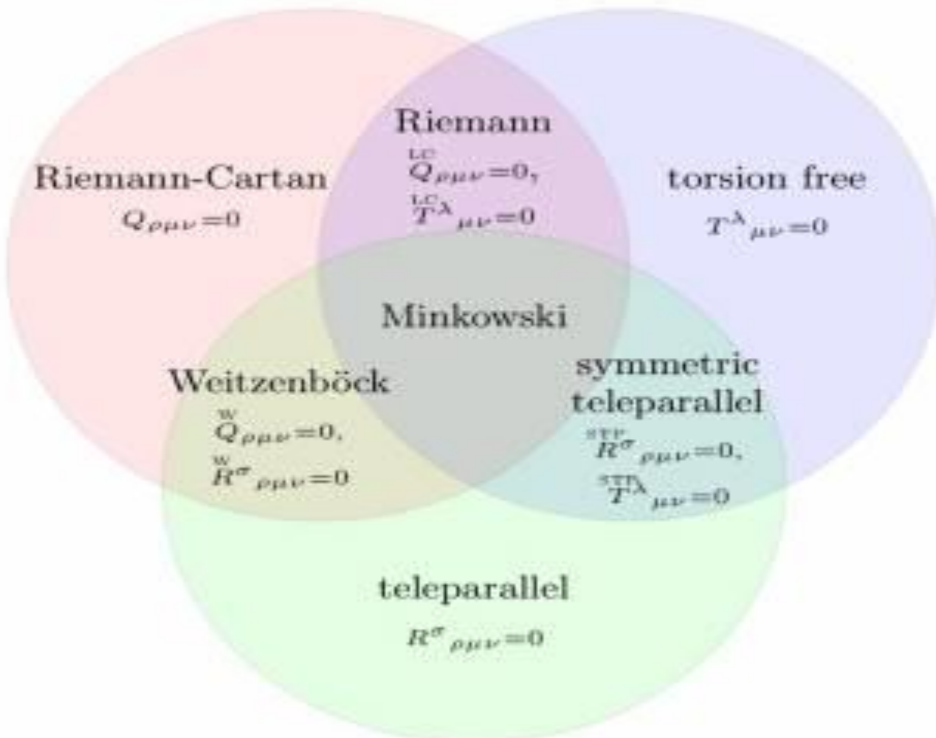


FIG. 1. Subclasses of metric-affine geometry, depending on the properties of connection.

$$S_{\rm GR} = \frac{1}{2\kappa^2}\int \left\{ g^{\mu\nu}\hat{R}_{\mu\nu} + \lambda_{(1)}^{\mu\nu\lambda} T_{\mu\nu\lambda} + \lambda_{(2)}^{\mu\nu\lambda} Q_{\mu\nu\lambda} \right\}\sqrt{-g}\, d^4x\,,$$
$$S_{\rm total} = S_{\rm GR} + S_{\rm matter}\,,$$

Curvature and Torsion

The dynamical variables in torsional formulation of gravity are the vielbein field $e_a(x^\mu)$, and the connection 1-forms $\omega^a{}_b(x^\mu)$ which defines the parallel transportation. In terms of coordinates, they can be expressed in components as $e_a = e_a{}^\mu \partial_\mu$ and $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu = \omega^a{}_{bc} e^c$. The dual vielbein is defined as $e^a = e^a{}_\mu dx^\mu$. One can express the commutation relations of the vielbein as

$$[e_a, e_b] = C^c{}_{ab} e_c \,, \tag{1}$$

where $C^c{}_{ab}$ are the structure coefficients functions given by

$$C^c{}_{ab} = e_a{}^\mu e_b{}^\nu (e^c{}_{\mu,\nu} - e^c{}_{\nu,\mu}) \,, \tag{2}$$

and comma denotes differentiation.

One can now define the torsion tensor, expressed in tangent components as

$$T^a{}_{bc} = \omega^a{}_{cb} - \omega^a{}_{bc} - C^a{}_{bc}\,, \tag{3}$$

and in "mixed" ones as

$$T^a{}_{\mu\nu} = e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu} + \omega^a{}_{b\mu}e^b{}_\nu - \omega^a{}_{b\nu}e^b{}_\mu\,. \tag{4}$$

Similarly, one can define the curvature tensor as

$$\begin{aligned} R^a{}_{bcd} &= \omega^a{}_{bd,c} - \omega^a{}_{bc,d} + \omega^e{}_{bd}\omega^a{}_{ec} - \omega^e{}_{bc}\omega^a{}_{ed} - C^e{}_{cd}\omega^a{}_{be} \\ R^a{}_{b\mu\nu} &= \omega^a{}_{b\nu,\mu} - \omega^a{}_{b\mu,\nu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu}\,. \end{aligned} \tag{5}$$

Thus, as one can see from (4) and (5), although the torsion tensor depends on both the vielbein and the connection, that is $T^a{}_{\mu\nu}(e^a{}_\mu, \omega^a{}_{b\mu})$, the curvature tensor depends only on the connection, namely $R^a{}_{b\mu\nu}(\omega^a{}_{b\mu})$.

Additionally, there is an independent object which is the metric tensor g. This allows us to make the vielbein orthonormal $g(e_a, e_b) = \eta_{ab}$, where $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots 1)$, and we have the relation

$$g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu. \tag{6}$$

Curvature and Torsion

- Vierbeins e_A^{μ} : four linearly independent fields in the tangent space
  $g_{\mu\nu}(x) = \eta_{AB}\ e_{\mu}^{A}(x)\ e_{\nu}^{B}(x)$
- Connection: ω_{ABC}
- Curvature tensor: $R^{A}_{B\mu\nu} = \omega^{A}_{B\nu,\mu} - \omega^{A}_{B\mu,\nu} + \omega^{A}_{C\mu}\omega^{C}_{B\nu} - \omega^{A}_{C\nu}\omega^{C}_{B\mu}$
- Torsion tensor: $T^{A}_{\mu\nu} = e^{A}_{\nu,\mu} - e^{A}_{\mu,\nu} + \omega^{A}_{B\mu}e^{B}_{\nu} - \omega^{A}_{B\nu}e^{B}_{\mu}$

Curvature and Torsion

- Vierbeins e^{μ}_{A} : four linearly independent fields in the tangent space

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- Levi-Civita connection and Contortion tensor: $\omega_{ABC} = \Gamma_{ABC} + K_{ABC}$

  $$K_{ABC} = \frac{1}{2}\left(T_{CAB} - T_{BCA} - T_{ABC}\right) = -K_{BAC}$$

- Curvature and Torsion Scalars: $\boxed{R = \overline{R} + T - 2\left(T^{\nu\mu}_{\nu}\right)_{;\mu}}$

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\rho}_{\mu\rho\nu}$$

$$T = \frac{1}{4}T^{\rho\mu\nu}T_{\rho\mu\nu} + \frac{1}{2}T^{\rho\mu\nu}T_{\nu\mu\rho} - \mathrm{T}^{\rho}_{\rho\mu}\mathrm{T}^{\nu\mu}_{\nu}$$

Additional motivation

- **Gauge Principle**: global symmetries replaced by local ones:

  The group generators give rise to the compensating fields

  It works perfect for the standard model of strong, weak and E/M interactions

$$SU(3) \times SU(2) \times U(1)$$

- Can we apply this to gravity?

Additional motivation

- Formulating the gauge theory of gravity
  (mainly after 1960):
- Start from Special Relativity

$\Longrightarrow$ Apply (Weyl-Yang-Mills) gauge principle to its Poincaré-group symmetries

$\Longrightarrow$ Get Poinaré gauge theory:
Both curvature and torsion appear as field strengths

- Torsion is the field strength of the translational group
  (Teleparallel and Einstein-Cartan theories are subcases of Poincaré theory)

Additional motivation

- One could extend the gravity gauge group (SUSY, conformal, scale, metric affine transformations)

  obtaining SUGRA, conformal, Weyl, metric affine gauge theories of gravity

- In all of them torsion is always related to the gauge structure.
- Thus, a possible way towards gravity quantization would need to bring torsion into gravity description.

Additional motivation

- 1998: Universe acceleration

$\Rightarrow$Thousands of work in Modified Gravity

(f(R), Gauss-Bonnet, Lovelock, nonminimal scalar coupling, nonminimal derivative coupling, Galileons, Hordenski, massive etc)

- Almost all in the curvature-based formulation of gravity

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- Almost all in the curvature-based formulation of gravity

- So question: Can we modify gravity starting from its torsion-based formulation?

torsion $\Rightarrow$ gauge $? \Rightarrow$ quantization

modification $\Rightarrow$ full theory $? \Rightarrow$ quantization

Teleparallel Equivalent of General Relativity (TEGR)

- Let's start from the simplest tosion-based gravity formulation, namely TEGR:
- Vierbeins e_A^{μ} : four linearly independent fields in the tangent space

$$g_{\mu\nu}(x) = \eta_{AB}\, e_{\mu}^{A}(x)\, e_{\nu}^{B}(x)$$

- Use curvature-less Weitzenböck connection instead of torsion-less Levi-Civita one: $\Gamma_{\nu\mu}^{\lambda\{W\}} = e_A^{\lambda}\partial_{\mu}e_{\nu}^{A}$
- Torsion tensor:

$$T_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda\{W\}} - \Gamma_{\mu\nu}^{\lambda\{W\}} = e_A^{\lambda}\left(\partial_{\mu}e_{\nu}^{A} - \partial_{\nu}e_{\mu}^{A}\right)$$

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$$T_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda\{W\}} - \Gamma_{\mu\nu}^{\lambda\{W\}} = e_A^\lambda\left(\partial_\mu e_\nu^A - \partial_\nu e_\mu^A\right)$$

- Lagrangian (imposing coordinate, Lorentz, parity invariance, and up to 2nd order in torsion tensor)

$$\boxed{L \equiv T = \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu}^{\rho} T_{\nu}^{\nu\mu}}$$

- **Completely equivalent** with GR at the level of equations

f(T) Gravity and f(T) Cosmology

- f(T) Gravity: Simplest torsion-based modified gravity
- Generalize T to f(T) (inspired by f(R))

$$S=\frac{1}{16\pi G}\int d^4x\, e\,\left[T+f(T)\right]+S_m$$

- Equations of motion:

$$e^{-1}\partial_\mu\left(e e_A^\rho S_\rho^{\ \mu\nu}\right)\left(1+f_T\right)-e_A^\lambda T^\rho_{\ \mu\lambda}S_\rho^{\ \nu\mu}+e_A^\rho S_\rho^{\ \mu\nu}\partial_\mu(T)\,f_{TT}-\frac{1}{4}e_A^\nu[T+f(T)]=4\pi G e_A^\rho \mathrm{T}_\rho^{\ \nu\{\mathrm{EM}\}}$$

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- f(T) Cosmology: Apply in FRW geometry:

$$e_\mu^A=diag(1,a,a,a)\ \Rightarrow\ ds^2=dt^2-a^2(t)\delta_{ij}dx^i dx^j$$ (not unique choice)

- Friedmann equations:

$$\boxed{H^2=\frac{8\pi G}{3}\rho_m-\frac{f(T)}{6}-2f_T H^2}$$

$$\boxed{\dot{H}=-\frac{4\pi G(\rho_m+p_m)}{1+f_T-12H^2 f_{TT}}}$$

- Find easily

$$T=-6H^2$$

f(T) Cosmology: Background

- Effective Dark Energy sector:

$$\rho_{DE} \equiv \frac{3}{8\pi G_N}\left[-\frac{f}{6}+\frac{Tf_T}{3}\right],$$

$$P_{DE} \equiv \frac{1}{16\pi G_N}\left[\frac{f-f_T T+2T^2 f_{TT}}{1+f_T+2Tf_{TT}}\right]$$

$$w_{DE} = -\frac{f - Tf_T + 2T^2 f_{TT}}{[1+f_T+2Tf_{TT}][f-2Tf_T]}$$

[Linder PRD 82]

- Interesting cosmological behavior: Late-time acceleration, Inflation etc

[Cai, Capozziello, De Laurentis, Saridakis, Rept.Prog.Phys. 79]

f(T) Cosmology: Background

- Re-write Firedmann Equation as:

$$E^2(z, \mathbf{r}) = \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{F0}\, y(z, \mathbf{r})$$

with $E^2(z) \equiv \frac{H^2(z)}{H_0^2} = \frac{T(z)}{T_0}$ and $\Omega_{F0} = 1 - \Omega_{m0} - \Omega_{r0}$;

- $\boxed{y(z, \mathbf{r}) = \frac{1}{T_0 \Omega_{F0}} \left[f - 2T f_T\right]}$ quantifies the deviation from ΛCDM
(for f=const. we obtain ΛCDM)

f(T) Cosmology: Perturbations

- For scalar perturbations:

$$e_\mu^0 = \delta_\mu^0(1+\psi)\ ,\ e_\mu^\alpha = \delta_\mu^\alpha a(1-\phi)$$

$$\Rightarrow\ ds^2 = (1+2\psi)dt^2 - a^2(1-2\phi)\delta_{ij}dx^i dx^j$$

- Obtain Perturbation Equations.

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\rm eff}\,\rho\,\delta \approx 0$$

$$Q(a) = \frac{G_{\rm eff}(a)}{G_N} = \frac{1}{1+f_T}$$

[Chen, Dent, Dutta, Saridakis PRD 83], [Dent, Dutta, Saridakis JCAP 1101]

$$\begin{aligned}
\delta \overset{\mathbf{em}}{T}{}_0{}^0 &= -\delta\rho_m \\
\delta \overset{\mathbf{em}}{T}{}_0{}^i &= a^{-2}(\rho_m+p_m)(-\partial_i\delta u) \\
\delta \overset{\mathbf{em}}{T}{}_i{}^0 &= (\rho_m+p_m)(\partial_i\delta u) \\
\delta \overset{\mathbf{em}}{T}{}_i{}^j &= \delta_{ij}\delta p_m + \partial_i\partial_j\pi^S .
\end{aligned}$$

$$\begin{aligned}
E_0^0 \equiv\ & (1+f_0')(\nabla^2\phi) + 6(1+f_0')H\dot{\phi} \\
& +6(1+f_0')H^2\psi - 3f_1'H^2 \\
& -\frac{T_1+f_1}{4} = -4\pi G\delta\rho_m, \\
E_0^i \equiv\ & (1+f_0')\partial_i\dot{\phi} + (1+f_0')H\partial_i\psi \\
& -12H\dot{H}f_0''\partial_i\phi = -4\pi G(\rho_m+p_m)\partial_i\delta u, \\
E_a^0 \equiv\ & 12H^2\partial_i\delta_a^i(\dot{\phi}+H\psi)f_0'' - (1+f_0')\partial_i\delta_a^i(\dot{\phi}+H\psi) \\
& = 4\pi G(\rho_m+p_m)\partial_i\delta_a^i\delta u,
\end{aligned}$$

$$\begin{aligned}
E_a^i\delta_i^a \equiv\ & \frac{f_1'}{a}\left(-3H^2-\dot{H}\right) + \frac{f_1''}{a}\left(12H^2\dot{H}\right) \\
& -\frac{(1+f_0')}{2a}\sum_{b\neq a}\partial^j\delta_j^b\partial_i\delta_b^i(\psi-\phi) \\
& -\frac{\phi(T_0+f_0)}{4a} - \frac{T_1+f_1}{4a} \\
& +\frac{(1+f_0')}{a}[6H\dot{\phi} + 6H^2\psi - 3H^2\phi \\
& \qquad +\ddot{\phi} + \dot{H}(2\psi-\phi) + H\dot{\psi}] \\
& +\frac{f_0''}{a}(-24H\dot{H}\dot{\phi} - 48\psi H^2\dot{H} - 12H^2\ddot{\phi} \\
& \qquad -12H^3\dot{\psi} + 12H^2\dot{H}\phi) \\
& \qquad = \frac{4\pi G}{a}(p_m\phi + \delta p_m), \\
E^i_{b;\,b\neq a}\delta_i^a \equiv\ & \frac{(1+f_0')}{2}\partial_j\delta_b^j\partial^i\delta_i^a(\phi-\psi) \\
& = 4\pi G a^2\partial_j\delta_b^j\partial^i\delta_i^a\pi^S
\end{aligned}$$

Viable f(T) models

- 1) Power-law model (f1CDM)

$$\boxed{f(T) = \alpha(-T)^b} \qquad \alpha = (6H_0^2)^{1-b}\frac{\Omega_{F0}}{2b-1}$$

$$y(z,b) = E^{2b}(z,b) \qquad G_{\mathrm{eff}}(z) = \frac{G_N}{1+\frac{b\Omega_{F0}}{(1-2b)E^{2(1-b)}}}$$

- 2) The Linder model (f2CDM)

$$\boxed{f(T) = \alpha T_0(1-e^{-p\sqrt{T/T_0}})} \qquad \alpha = \frac{\Omega_{F0}}{1-(1+p)e^{-p}}$$

$$y(z,p) = \frac{1-(1+pE)e^{-pE}}{1-(1+p)e^{-p}} \qquad G_{\mathrm{eff}}(z) = \frac{G_N}{1+\frac{\Omega_{F0}p\ e^{-pE}}{2E[1-(1+p)e^{-p}]}}$$

- 3) The exponential model (f3CDM)

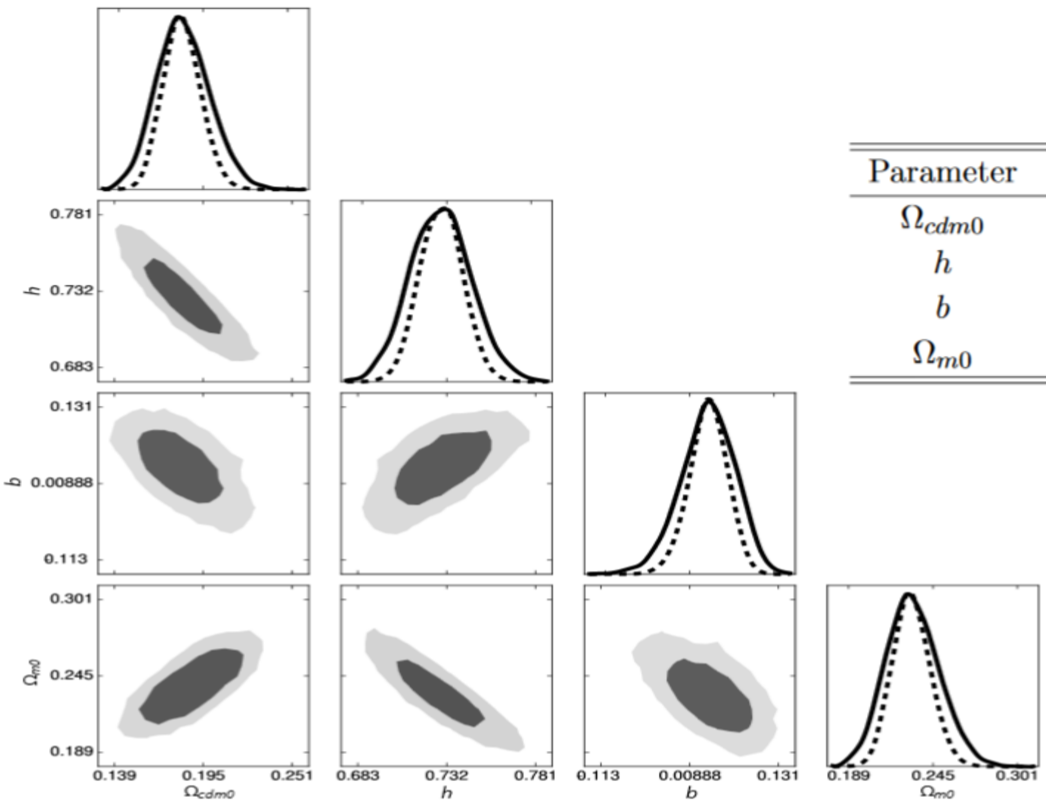
$$\boxed{f(T) = \alpha T_0(1-e^{-pT/T_0})} \qquad \alpha = \frac{\Omega_{F0}}{1-(1+2p)e^{-p}}$$

$$y(z,p) = \frac{1-(1+2pE^2)e^{-pE^2}}{1-(1+2p)e^{-p}} \qquad G_{\mathrm{eff}}(z) = \frac{G_N}{1+\frac{\Omega_{F0}p\ e^{-pE^2}}{1-(1+2p)e^{-p}}}$$

[Nesseris, Basilakos, Saridakis, Perivolaropoulos, PRD 88]

Viable f(T) models

- Power-law (f1CDM): $f(T) = \alpha(-T)^b$ $\alpha = (6H_0^2)^{1-b}\dfrac{\Omega_{F0}}{2b-1}$



Parameter	best-fit	mean$\pm$ 1σ	95% lower	95% upper
Ω_{cdm0}	0.1806	$0.1835^{+0.016}_{-0.019}$	0.1503	0.2179
h	0.7292	$0.7275^{+0.017}_{-0.018}$	0.6945	0.7616
b	0.05536	$0.05128^{+0.025}_{-0.019}$	0.00622	0.09329
Ω_{m0}	0.2306	$0.2335^{+0.016}_{-0.019}$	0.2003	0.2679

SNIa, BAO, CC

[Nunes, Pan, Saridakis, JCAP 1608]

[Basilakos, Nesseris, Anagnostopoulos, Saridakis, 1803.09278]

Non-minimally coupled scalar-torsion theory

- In curvature-based gravity, apart from $R+f(R)$ one can use $R+\xi R\varphi$^2
- Let's do the same in torsion-based gravity:

$$\boxed{S=\int d^4x\, e\left[\frac{T}{2\kappa^2}+\frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi+\xi T\varphi^2\right)-V(\varphi)+L_m\right]}$$

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- Friedmann equations in FRW universe:

$$H^2=\frac{\kappa^2}{3}(\rho_m+\rho_{DE})$$

$$\dot{H}=-\frac{\kappa^2}{2}(\rho_m+p_m+\rho_{DE}+p_{DE})$$

with effective Dark Energy sector: $$\rho_{DE}=\frac{\dot{\varphi}^2}{2}+V(\varphi)-3\xi H^2\varphi^2$$

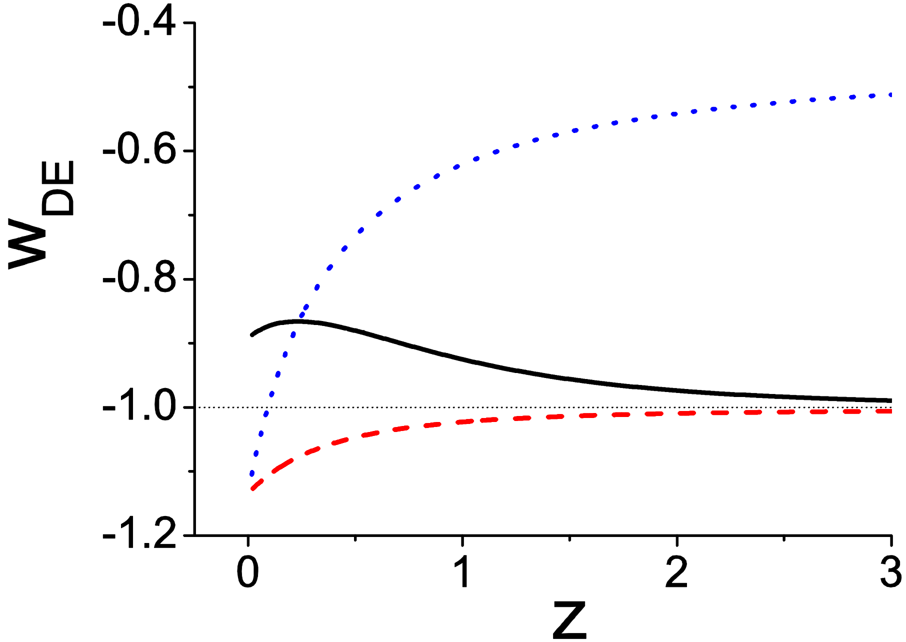
$$p_{DE}=\frac{\dot{\varphi}^2}{2}-V(\varphi)+4\xi H\varphi\dot{\varphi}+\xi\left(3H^2+2\dot{H}\right)\varphi^2$$

- **Different** than non-minimal quintessence!

(no conformal transformation in the present case)

Non-minimally coupled scalar-torsion theory

- Main advantage: Dark Energy may lie in the phantom regime or/and experience the phantom-divide crossing
- Teleparallel Dark Energy:



Non-minimally matter-torsion coupled theory

- In curvature-based gravity, one can use $f(R)L_m$ coupling
- Let's do the same in torsion-based gravity:

$$\boxed{S=\frac{1}{2\kappa^2}\int d^4x\, e\left\{T+f_1(T)+\left[1+\lambda\, f_2(T)\right]L_m\right\}}$$

Non-minimally matter-torsion coupled theory

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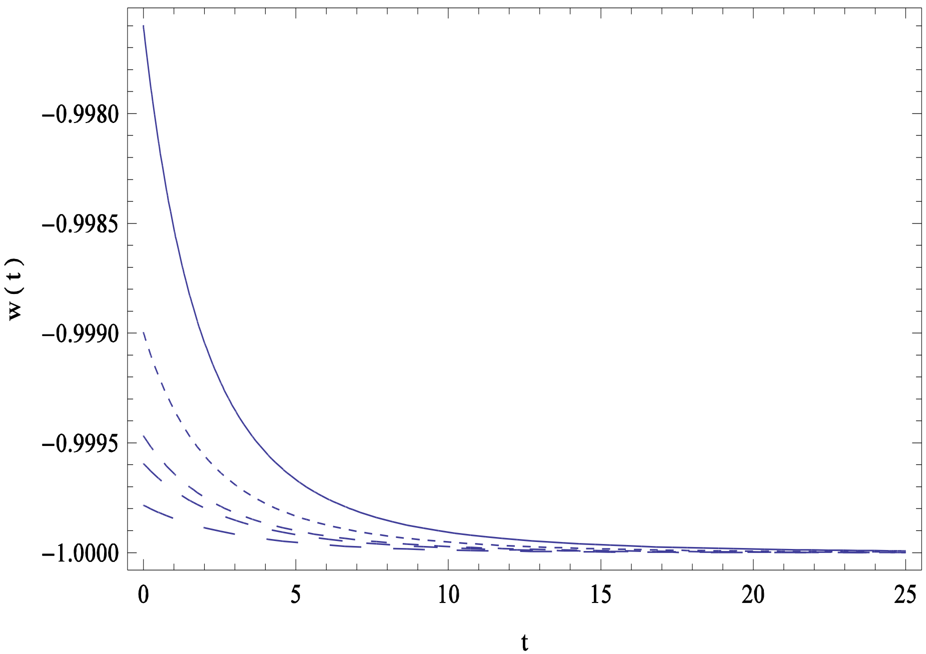
with effective Dark Energy sector: $\rho_{DE}=-\frac{1}{2\kappa^2}\left(f_1+12H^2 f_1'\right)+\lambda\rho_m\left(f_2+12H^2 f_2'\right)$

$$p_{DE}=(\rho_m+p_m)\left[\frac{1+\lambda\left(f_2+12H^2 f_2'\right)}{1+f_1'-12H^2 f_1''-2\kappa^2\lambda\rho_m\left(f_2'-12H^2 f_2''\right)}\right]+\frac{\lambda\left(f_1+12H^2 f_1'\right)}{2\kappa^2}-\lambda\rho_m\left(f_2+12H^2 f_2'\right)$$

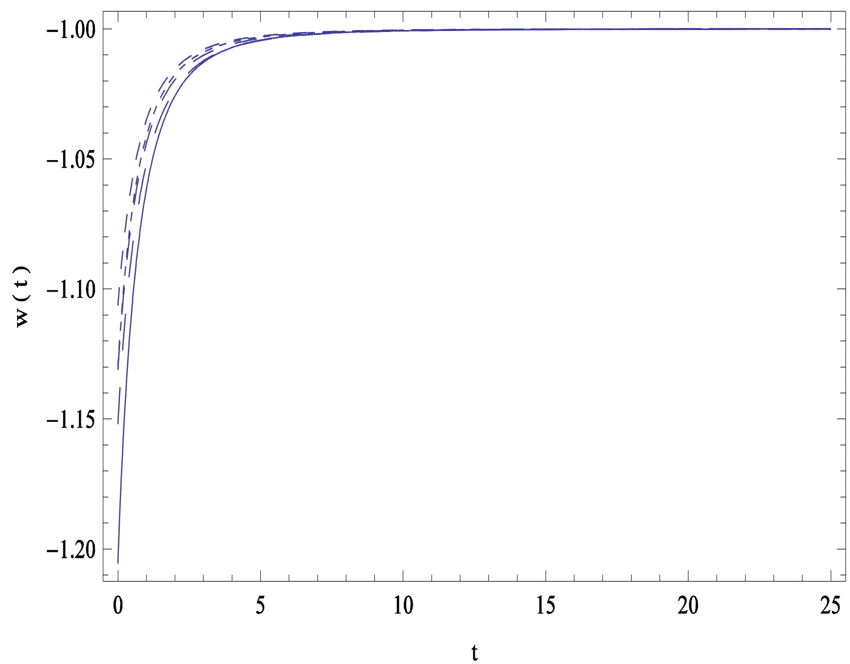
- **Different** than non-minimal matter-curvature coupled theory

Non-minimally matter-torsion coupled theory

- Interesting phenomenology



$$f_1(T) = -\Lambda + \alpha_1 T^2, \quad f_2(T) = \beta_1 T^2$$



$$f_1(T) = -\Lambda, \quad f_2(T) = \alpha_1 T + \beta_1 T^2$$

Teleparallel Equivalent of Gauss-Bonnet and f(T,T_G) gravity

- In curvature-based gravity, one can use higher-order invariants like the Gauss-Bonnet one $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$
- Let's do the same in torsion-based gravity:
- Similar to $e\overline{R} = -eT + 2\left(eT_{\nu}^{\nu\mu}\right)_{,\mu}$ we construct $e\overline{G} = eT_G + tot.diverg$ with

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- Let's do the same in torsion-based gravity:
- Similar to $e\bar{R} = -eT + 2\left(eT_{\nu}^{\nu\mu}\right)_{,\mu}$ we construct $e\bar{G} = eT_G + tot.diverg$ with

$$T_G = \left(K^{a_1}_{\ \ ea_2} K^{ea_2}_{\ \ b} K^{a_3}_{\ \ fc} K^{fa_4}_{\ \ d} - 2K^{a_1a_2}_{\ \ \ a} K^{a_3}_{\ \ eb} K^{e}_{\ fc} K^{fa_4}_{\ \ d} + 2K^{a_1a_2}_{\ \ \ a} K^{a_3}_{\ \ eb} K^{ea_4}_{\ \ f} K^{f}_{\ cd} + +2K^{a_1a_2}_{\ \ \ a} K^{a_3}_{\ \ eb} K^{ea_4}_{\ \ f} K^{f}_{\ c,d}\right)\delta^{abcd}_{a_1a_2a_3a_4}$$

- $f(\mathrm{T}, T_G)$ gravity:

$$S = \frac{1}{2\kappa^2}\int d^4x\, e\,\{T + f(T, T_G)\} + S_m$$

- **Different** from $f(R,G)$ and $f(T)$ gravities

Teleparallel Equivalent of Gauss-Bonnet and f(T,T_G) gravity

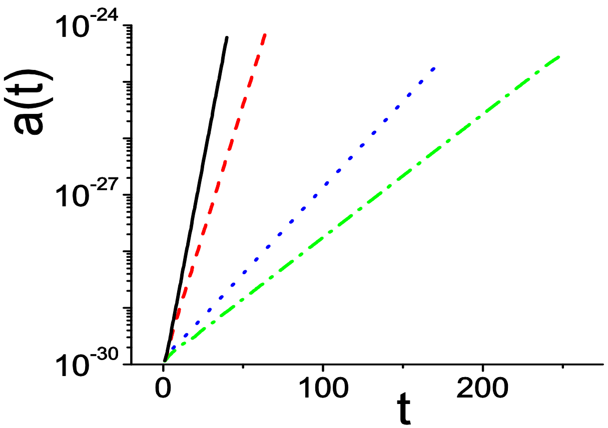
- Cosmological application:

$$\rho_{DE} = -\frac{1}{2\kappa^2}\left[f - 12H^2 f_T - T_G f_{T_G} + 24H^3 \dot{f}_{T_G}\right]$$

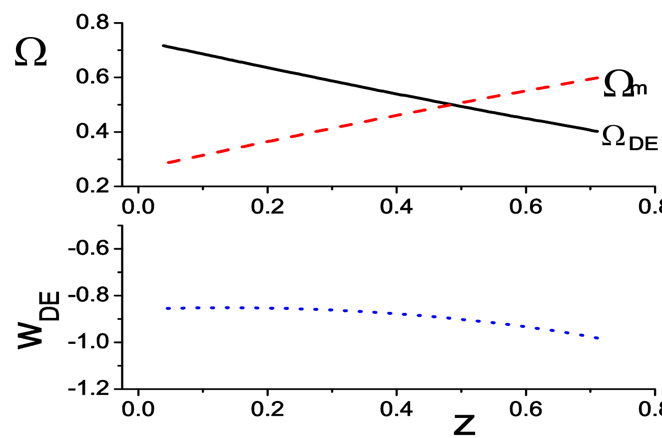
$$T = 6H^2$$

$$T_G = 24H^2\left(\dot{H} + H^2\right)$$

$$p_{DE} = \frac{1}{2\kappa^2}\left[f - 4\left(\dot{H} + 3H^2\right)f_T - 4H\dot{f}_T - T_G f_{T_G} + \frac{2}{3H}T_G \dot{f}_{T_G} + 8H^2 \ddot{f}_{T_G}\right]$$



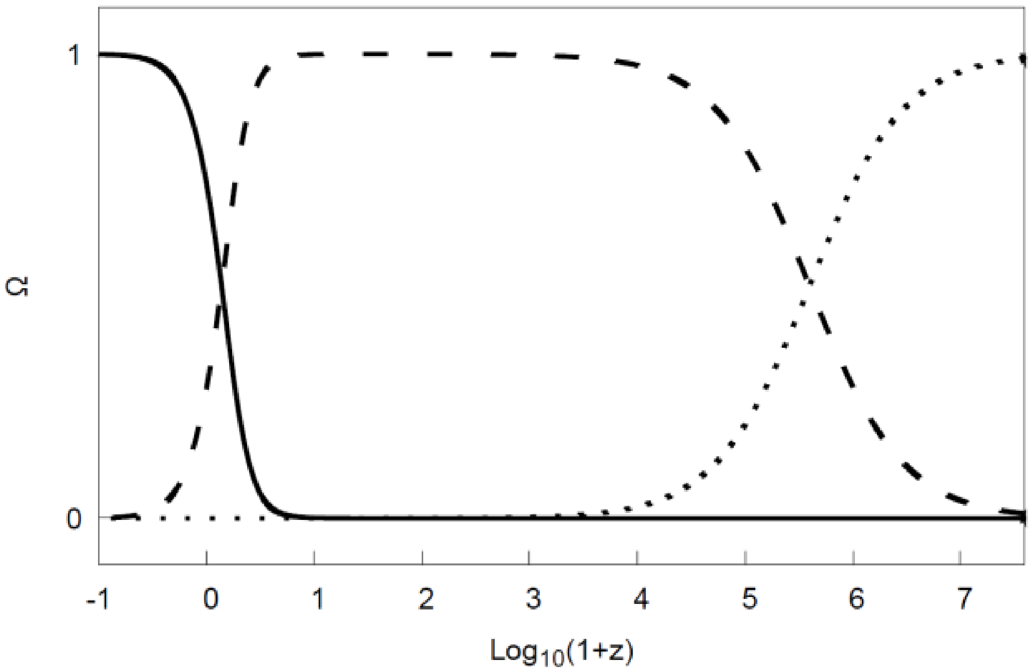
$$f(T,T_G) = \alpha_1 T^2 + \alpha_2 T\sqrt{|T_G|}$$

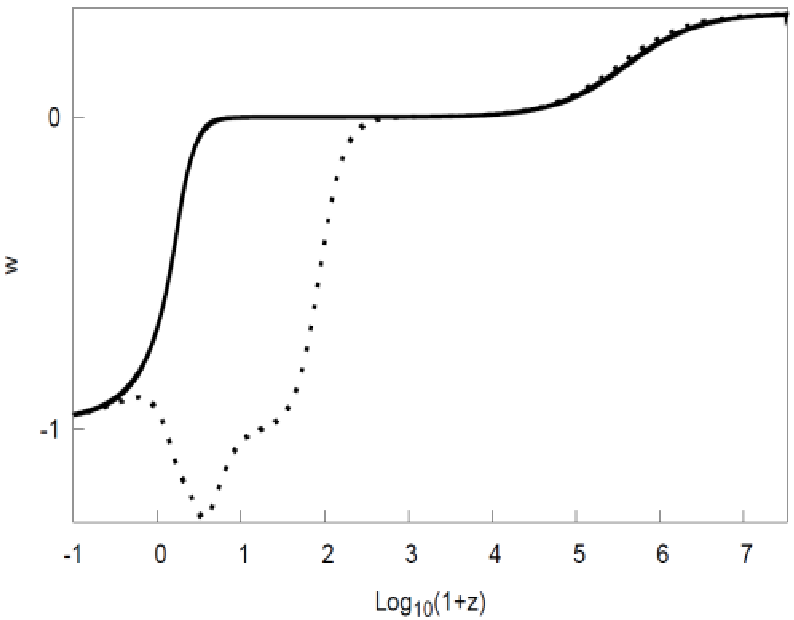


$$f(T,T_G) = \beta_1\sqrt{T^2 + \beta_2 T_G}$$

Torsional Gravity with higher derivatives

$$S = \frac{1}{2\kappa^2}\int d^4x\, e\, F\big(T,(\nabla T)^2,\Diamond T\big) + S_m(e^A_\mu,\Psi_m)$$





Torsional Modified Gravity

$$
\begin{tikzcd}
 & & f(T,B,T_G,B_G,\mathcal{T}) \arrow[dll, "f=f(-T+B,-T_G+B_G)"'] \arrow[d, "f=f(T,B,\mathcal{T})"] \arrow[drr, "f=f(T,T_G)"] & & \\
\boxed{f(R,G)} \arrow[ddd, "f=f(R)"'] & & f(T,B,\mathcal{T}) \arrow[dl, "f=f(-T+B,\mathcal{T})"'] \arrow[dr, "f=f(T,\mathcal{T})"] \arrow[ddd, "f=f(T,B)"] & & \boxed{f(T,T_G)} \arrow[ddd, "f=f(T)"] \\
 & f(R,\mathcal{T}) \arrow[ddl, "f=f(R)"'] & & f(T,\mathcal{T}) \arrow[ddr, "f=f(T)"] & \\
 & & & & \\
\boxed{f(R)} \arrow[drr, "f=R"'] & & f(T,B) \arrow[ll, "f=f(-T+B)"'] \arrow[rr, "f=f(T)"] & & \boxed{f(T)} \arrow[dll, "f=T"] \\
 & & \mathrm{GR}\ \&\ \mathrm{TEGR} & &
\end{tikzcd}
$$

[Bahamobde, Bohmer, EPJC 76]

Scalar-torsion theories – Teleparallel Horndeski

$$
\begin{aligned}
a_\mu &= \frac{1}{6}\epsilon_{\mu\nu\sigma\rho}T^{\nu\sigma\rho}\,,\\
v_\mu &= T^\sigma{}_{\sigma\mu}\,,\\
t_{\sigma\mu\nu} &= \frac{1}{2}\left(T_{\sigma\mu\nu}+T_{\mu\sigma\nu}\right)+\frac{1}{6}\left(g_{\nu\sigma}v_\mu+g_{\nu\mu}v_\sigma\right)-\frac{1}{3}g_{\sigma\mu}v_\nu\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{\rm ax} &= a_\mu a^\mu = \frac{1}{18}\left(T_{\sigma\mu\nu}T^{\sigma\mu\nu}-2T_{\sigma\mu\nu}T^{\mu\sigma\nu}\right)\,,\\
T_{\rm vec} &= v_\mu v^\mu = T^\sigma{}_{\sigma\mu}T_\rho{}^{\rho\mu}\,,\\
T_{\rm ten} &= t_{\sigma\mu\nu}t^{\sigma\mu\nu} = \frac{1}{2}\left(T_{\sigma\mu\nu}T^{\sigma\mu\nu}+T_{\sigma\mu\nu}T^{\mu\sigma\nu}\right)-\frac{1}{2}T^\sigma{}_{\sigma\mu}T_\rho{}^{\rho\mu}\,,
\end{aligned}
$$

$$
T=\frac{3}{2}T_{\rm ax}+\frac{2}{3}T_{\rm ten}-\frac{2}{3}T_{\rm vec}\,.
$$

$$
\begin{aligned}
I_1 &= t^{\mu\nu\sigma}\phi_{;\mu}\phi_{;\nu}\phi_{;\sigma}\,,\\
I_2 &= v^\mu\phi_{;\mu}\,,\\
I_3 &= a^\mu\phi_{;\mu}\,.
\end{aligned}
$$

$$
\begin{aligned}
J_1 &= a^\mu a^\nu\phi_{;\mu}\phi_{;\nu}\,,\\
J_2 &= v^\mu v^\nu\phi_{;\mu}\phi_{;\nu}\,,\\
J_3 &= v_\sigma t^{\sigma\mu\nu}\phi_{;\mu}\phi_{;\nu}\,,\\
J_4 &= v_\mu t^{\sigma\mu\nu}\phi_{;\sigma}\phi_{;\nu}\,,\\
J_5 &= t^{\sigma\mu\nu}t_\sigma{}^{\bar\mu}{}_\nu\phi_{;\mu}\phi_{;\bar\mu}\,,\\
J_6 &= t^{\sigma\mu\nu}t_\sigma{}^{\bar\mu\bar\nu}\phi_{;\mu}\phi_{;\nu}\phi_{;\bar\mu}\phi_{;\bar\nu}\,,\\
J_7 &= t^{\sigma\mu\nu}t^{\bar\sigma\bar\mu}{}_\sigma\phi_{;\mu}\phi_{;\nu}\phi_{;\bar\sigma}\phi_{;\bar\mu}\,,\\
J_8 &= t^{\sigma\mu\nu}t_{\sigma\mu}{}^{\bar\nu}\phi_{;\nu}\phi_{;\bar\nu}\,,\\
J_9 &= t^{\sigma\mu\nu}t^{\bar\sigma\bar\mu\bar\nu}\phi_{;\sigma}\phi_{;\mu}\phi_{;\nu}\phi_{;\bar\sigma}\phi_{;\bar\mu}\phi_{;\bar\nu}\,,\\
J_{10} &= \epsilon^\mu{}_{\nu\rho\sigma}a^\nu t^{\alpha\rho\sigma}\phi_{;\mu}\phi_{;\alpha}\,.
\end{aligned}
$$

$$
\mathcal{L}_{\rm Tele}=G_{\rm Tele}(\phi,X,T,T_{\rm ax},T_{\rm vec},I_2,J_1,J_3,J_5,J_6,J_8,J_{10}),
$$

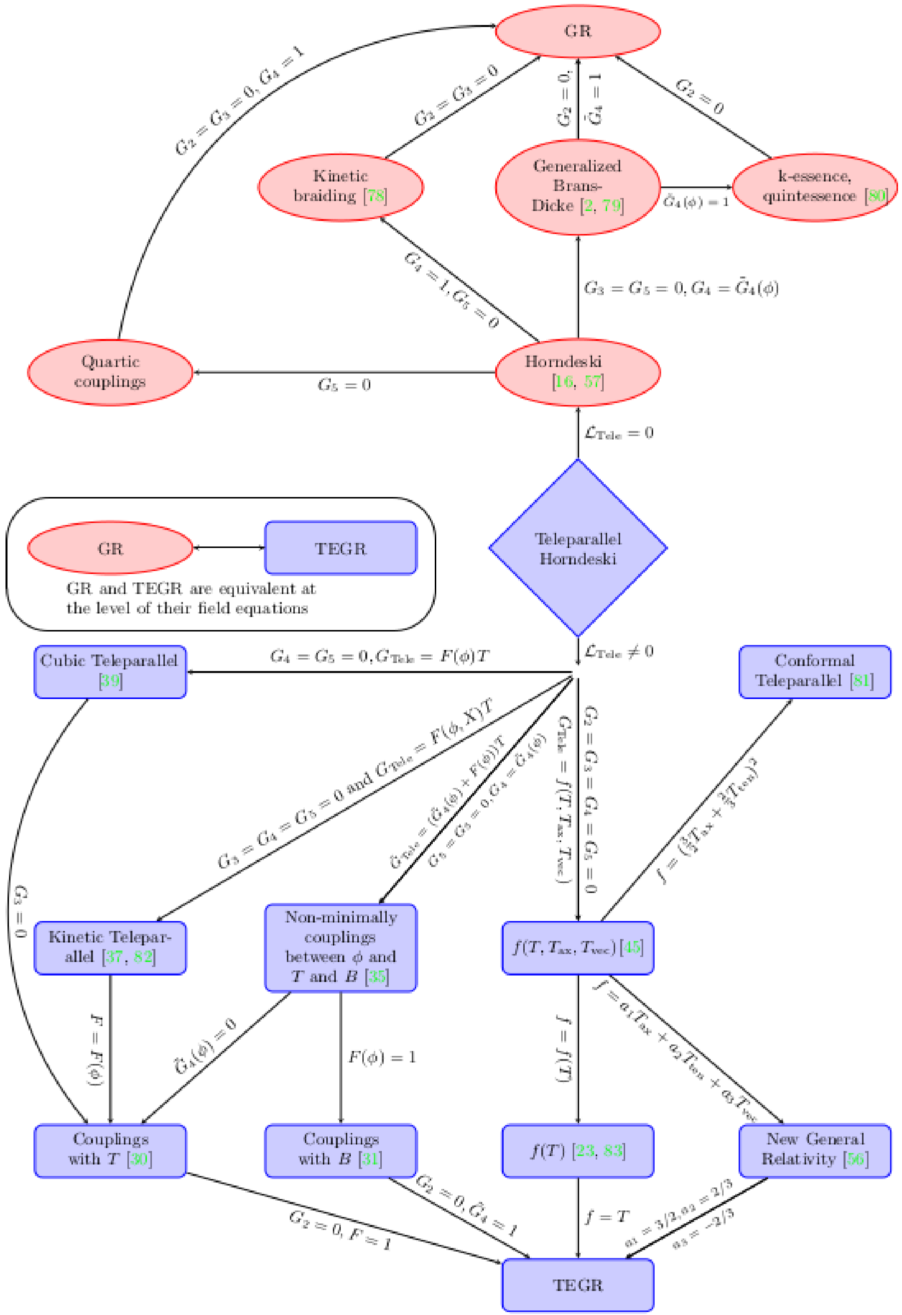


FIG. 1: Relationship between Teleparallel Horndeski and various theories.

Growth-index constraints on f(T) gravity

- Perturbations: $\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G_{eff}\,\rho_m \delta_m$, clustering growth rate: $\dfrac{d\ \ln \delta_m}{d\ \ln a} = \Omega_m^{\gamma}(a)$
- γ(z): Growth index. $G_{eff} = \dfrac{1}{1 + f'(T)}$

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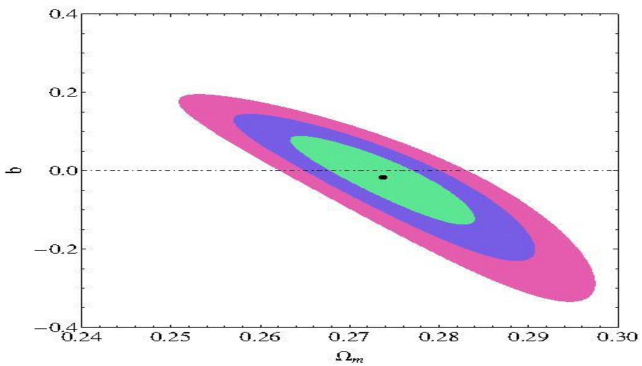
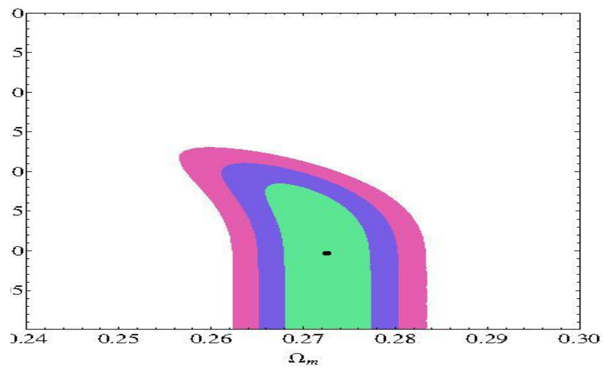
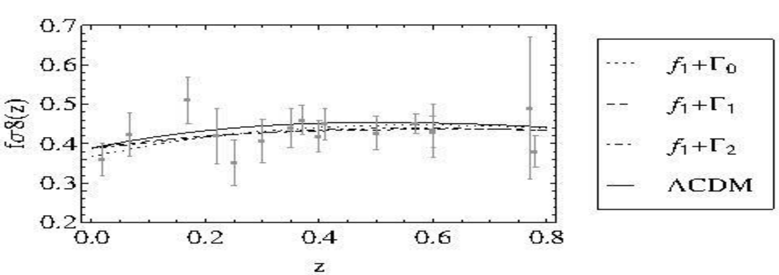
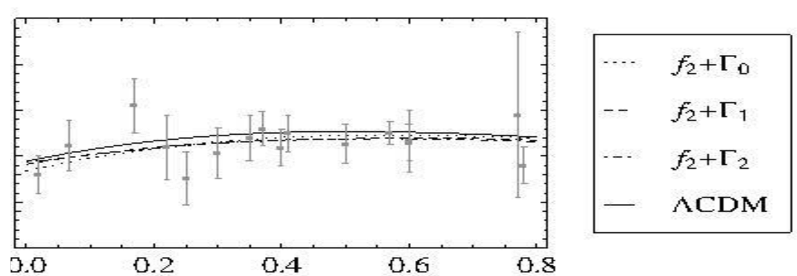
- γ(z): Growth index. $G_{eff} = \frac{1}{1+f'(T)}$



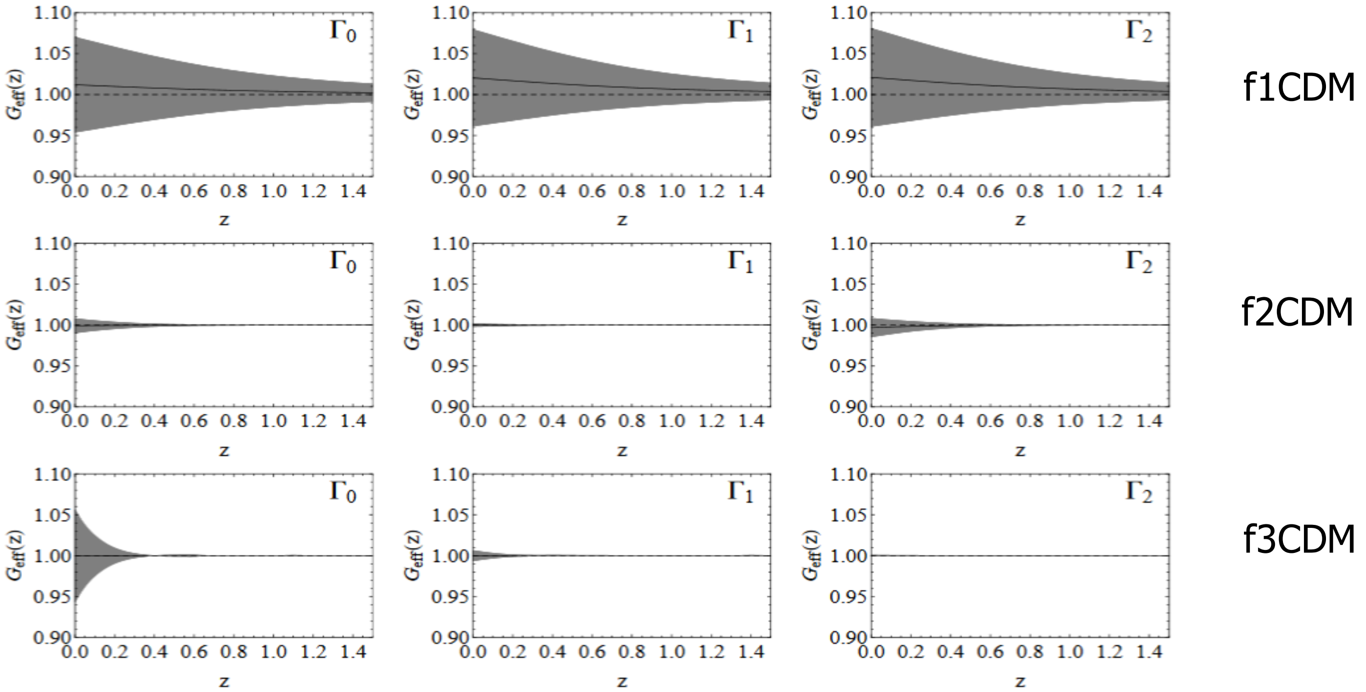






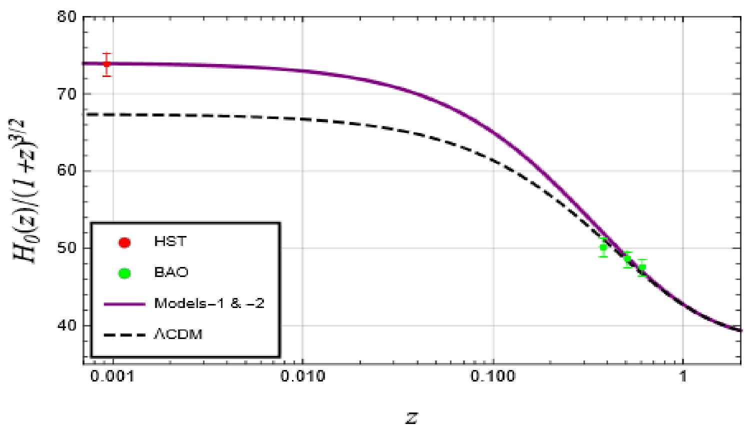
- Viable f(T) models are practically indistinguishable from ΛCDM.

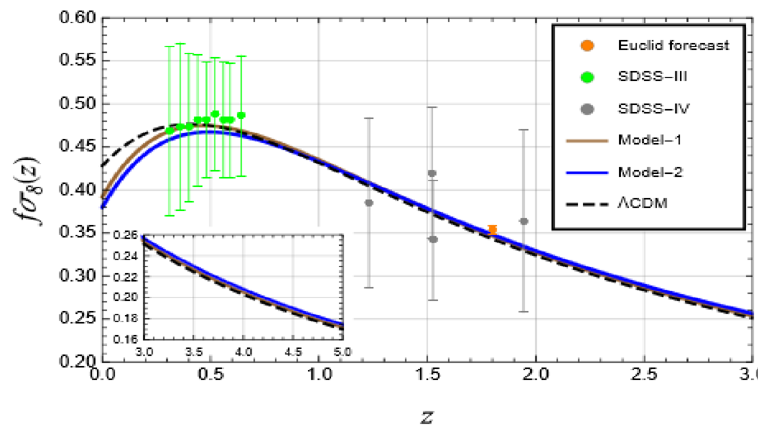
Viable f(T) models



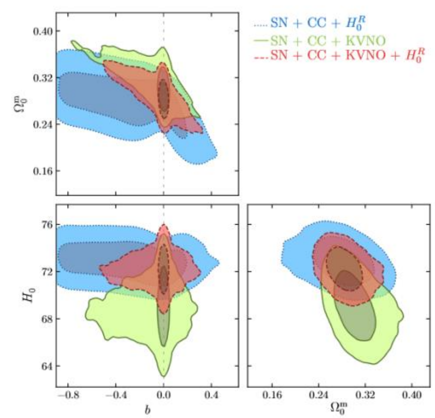
- In f(T) gravity we can indeed obtain $G_{\rm eff}/G_{\rm N}$ <1 for z<2, without affecting the background evolution.
- fσ8 tension may be alleviated. [Nesseris, Basilakos, Saridakis, Perivolaropoulos, PRD 88]

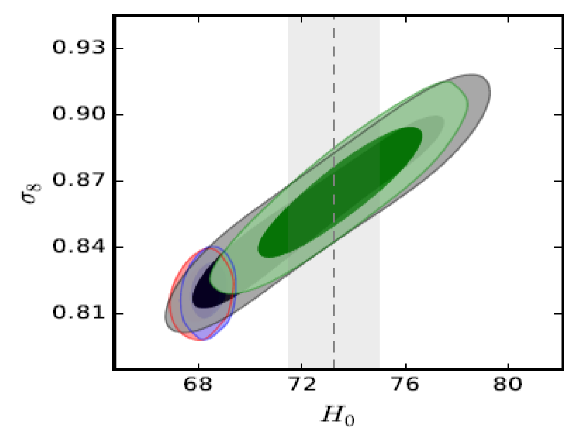
H0 and σ8 tension can be alleviated





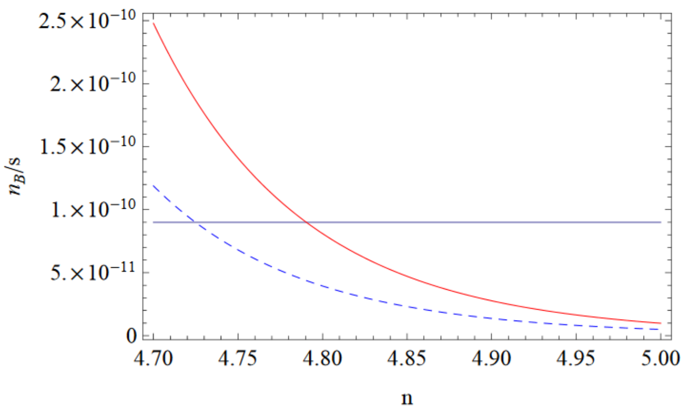
Parameter	CMB + BAO	CMB + BAO + H_0
$10^2\omega_b$	$2.235^{+0.013}_{-0.013}$	$2.235^{+0.013}_{-0.013}$
ω_{cdm}	$0.1181^{+0.001}_{-0.001}$	$0.118^{+0.001}_{-0.001}$
$100\theta_s$	$1.041^{+0.00027}_{-0.00027}$	$1.041^{+0.00030}_{-0.00027}$
$\ln 10^{10}A_s$	$3.078^{+0.023}_{-0.023}$	$3.08^{+0.022}_{-0.022}$
n_s	$0.9678^{+0.0039}_{-0.0039}$	$0.9684^{+0.0039}_{-0.0044}$
τ_{reio}	$0.073^{+0.012}_{-0.013}$	$0.075^{+0.012}_{-0.012}$
n	$0.0043^{+0.0033}_{-0.0039}$	$0.0054^{+0.0020}_{-0.0020}$
$\log\alpha$	$10.00^{+0.081}_{-0.12}$	$10.03^{+0.06}_{-0.06}$
Ω_{F0}	$0.73^{+0.021}_{-0.028}$	$0.738^{+0.015}_{-0.015}$
H_0	$72.4^{+3.3}_{-4.1}$	$73.5^{+2.1}_{-2.1}$
σ_8	$0.855^{+0.023}_{-0.033}$	$0.866^{+0.02}_{-0.02}$
$\chi^2_{min}/2$	6480.48	6482.27



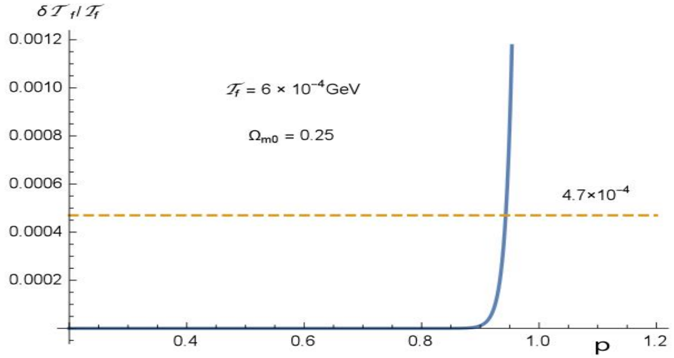


Baryogenesis and BBN constraints on f(T) gravity

- Baryon-anti-baryon asymmetry through CP violating term: $\frac{1}{M_*^2}\int d^4x\, e\left[\partial_\mu f(T)\right]J^\mu$



- BBN constraints: $\frac{\delta T_f}{T_f} \approx \frac{\rho_T}{\rho}\frac{H_{GR}}{10qT_f^5}$



Solar System constraints on f(T) gravity

- Apply the black hole solutions in Solar System:
- Assume corrections to TEGR of the form $\boxed{f(T)=\alpha T^2+O(T^3)}$

$$\Rightarrow F(r)^2=1-\frac{2GM}{c^2r}-\frac{\Lambda}{3}r^2+\alpha\left[-6\Lambda-\frac{6}{r^2}-\frac{4GM\Lambda}{c^2r}\right]$$

$$\Rightarrow G(r)^2=1-\frac{2GM}{c^2r}-\frac{\Lambda}{3}r^2+\alpha\left[\frac{8\Lambda}{3}-\frac{24}{r^2}-2\Lambda^2r^2-\frac{2GM}{c^2r}\left(8\Lambda-\frac{8}{r^2}\right)\right]$$

Solar System constraints on f(T) gravity

- Apply the black hole solutions in Solar System:
- Assume corrections to TEGR of the form $\boxed{f(T)=\alpha T^2+O(T^3)}$

$$\Rightarrow F(r)^2=1-\frac{2GM}{c^2r}-\frac{\Lambda}{3}r^2+\alpha\left[-6\Lambda-\frac{6}{r^2}-\frac{4GM\Lambda}{c^2r}\right]$$

$$\Rightarrow G(r)^2=1-\frac{2GM}{c^2r}-\frac{\Lambda}{3}r^2+\alpha\left[\frac{8\Lambda}{3}-\frac{24}{r^2}-2\Lambda^2r^2-\frac{2GM}{c^2r}\left(8\Lambda-\frac{8}{r^2}\right)\right]$$

- Use data from Solar System orbital motions:

$$\boxed{\Delta U_{f(T)}\leq 6.2\times 10^{-10}}$$

T<<1 so consistent

- f(T) divergence from TEGR is very small
- This was already known from cosmological observation constraints up to $O(10^{-1}-10^{-2})$
- With Solar System constraints, much more stringent bound.

f(Q) gravity

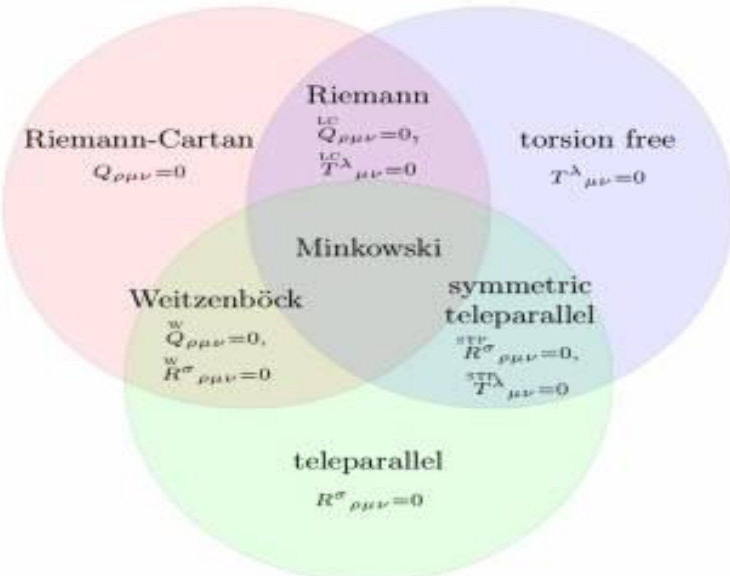


FIG. 1. Subclasses of metric-affine geometry, depending on the properties of connection.

affine connection $\Gamma^{\alpha}_{\mu\nu}$ can be decomposed as

$$\Gamma^{\alpha}_{\mu\nu} = \hat{\Gamma}^{\alpha}_{\mu\nu} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}, \tag{1}$$

where $\hat{\Gamma}^{\alpha}_{\mu\nu}$ is the Levi-Civita connection,

$$K^{\alpha}{}_{\mu\nu} = \frac{1}{2} T^{\alpha}{}_{\mu\nu} + T_{(\mu}{}^{\alpha}{}_{\nu)} \tag{2}$$

is the contortion tensor with $T^{\alpha}{}_{\mu\nu}$ the torsion tensor, and

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2} Q^{\alpha}{}_{\mu\nu} - Q_{(\mu}{}^{\alpha}{}_{\nu)} \tag{3}$$

is the disformation tensor arising from the non-metricity

$$Q_{\alpha\mu\nu} \equiv \nabla_{\alpha} g_{\mu\nu}, \tag{4}$$

f(Q) gravity

$$
\begin{aligned}
T^{\lambda}{}_{\mu\nu} &\equiv \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} \\
R^{\sigma}{}_{\rho\mu\nu} &\equiv \partial_{\mu}\Gamma^{\sigma}{}_{\nu\rho} - \partial_{\nu}\Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\sigma}{}_{\mu\alpha} - \Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\sigma}{}_{\nu\alpha}
\end{aligned}
\tag{5}
$$

while the nonmetricity can be expressed as

$$
Q_{\rho\mu\nu} \equiv \nabla_{\rho} g_{\mu\nu} = \partial_{\rho} g_{\mu\nu} - \Gamma^{\beta}{}_{\rho\mu} g_{\beta\nu} - \Gamma^{\beta}{}_{\rho\nu} g_{\mu\beta} \,. \tag{6}
$$

$$
Q = -\frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\gamma\beta\alpha} + \frac{1}{4} Q_{\alpha} Q^{\alpha} - \frac{1}{2} Q_{\alpha} \tilde{Q}^{\alpha} \,, \tag{7}
$$

where $Q_{\alpha} \equiv Q_{\alpha}{}^{\mu}{}_{\mu}$, and $\tilde{Q}^{\alpha} \equiv Q_{\mu}{}^{\mu\alpha}$.

f(Q) gravity

$$S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}f(Q). \qquad (8)$$

$$\frac{2}{\sqrt{-g}}\nabla_\alpha\left\{\sqrt{-g}g_{\beta\nu}f_Q\left[-\frac{1}{2}L^{\alpha\mu\beta}+\frac{1}{4}g^{\mu\beta}\left(Q^\alpha-\tilde{Q}^\alpha\right)\right.\right.$$
$$\left.\left.-\frac{1}{8}\left(g^{\alpha\mu}Q^\beta+g^{\alpha\beta}Q^\mu\right)\right]\right\}$$
$$+f_Q\left[-\frac{1}{2}L^{\mu\alpha\beta}-\frac{1}{8}\left(g^{\mu\alpha}Q^\beta+g^{\mu\beta}Q^\alpha\right)\right.$$
$$\left.+\frac{1}{4}g^{\alpha\beta}\left(Q^\mu-\tilde{Q}^\mu\right)\right]Q_{\nu\alpha\beta}+\frac{1}{2}\delta^\mu_\nu f = T^\mu{}_\nu\,, \qquad (9)$$

with $f_Q = \partial f/\partial Q$.

f(Q) cosmology

- Background:

$$
\begin{aligned}
6f_Q H^2 - \frac{1}{2}f &= \rho_m, \\
(12H^2 f_{QQ} + f_Q)\dot{H} &= -\frac{1}{2}(\rho_m + p_m)\,. \qquad (11)
\end{aligned}
$$

$$
Q = 6H^2, \qquad (12)
$$

f(Q) cosmology

- Perturbations:

$$\begin{aligned}-a^2\delta\rho &= 6\left(f_Q+12a^{-2}\mathcal{H}^2 f_{QQ}\right)\mathcal{H}\left(\mathcal{H}\phi+\varphi'\right)+2f_Q k^2\psi \\ &\quad -2\left[f_Q+3a^{-2}f_{QQ}\left(\mathcal{H}'+\mathcal{H}^2\right)\right]\mathcal{H}k^2 B\,. \qquad (19)\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}a^2\left(\rho+p\right)v=\left[f_Q+3a^{-2}f_{QQ}\left(\mathcal{H}'+\mathcal{H}^2\right)\right]\mathcal{H}\phi \\ &+6a^{-2}f_{QQ}\mathcal{H}^2\varphi'-9a^{-2}f_{QQ}\left(\mathcal{H}'-\mathcal{H}^2\right)\mathcal{H}\varphi \\ &+f_Q\psi'-a^{-2}f_{QQ}\mathcal{H}^2k^2B\,, \qquad (20)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}a^2\delta p &= \left(f_Q+12a^{-2}f_{QQ}\mathcal{H}^2\right)\left(\mathcal{H}\phi'+\varphi''\right)+\left[f_Q\left(\mathcal{H}'+2\mathcal{H}^2-\frac{1}{3}k^2\right)+12a^{-2}f_{QQ}\mathcal{H}^2\left(4\mathcal{H}'-\mathcal{H}^2\right)+12a^{-2}\frac{df_{QQ}}{d\tau}\mathcal{H}^3\right]\phi \\ &+\ 2\left[f_Q+6a^{-2}f_{QQ}\left(3\mathcal{H}'-\mathcal{H}^2\right)+6a^{-2}\frac{df_{QQ}}{d\tau}\mathcal{H}\right]\mathcal{H}\varphi'+\frac{1}{3}f_Q k^2\psi \\ &-\ \frac{1}{3}\left(f_Q+6a^{-2}f_{QQ}\mathcal{H}^2\right)k^2B'-\frac{1}{3}\left[2f_Q+3a^{-2}f_{QQ}\left(5\mathcal{H}-\mathcal{H}^2\right)+6a^{-2}\frac{df_{QQ}}{d\tau}\mathcal{H}\right]\mathcal{H}k^2B\,, \qquad (21)\end{aligned}$$

f(Q) cosmology

- Perturbations:

$$\delta' = (1+w)\left(-k^2 v - k^2 B + 3\varphi'\right) + 3\mathcal{H}\left(w\rho - \frac{\delta p}{\rho}\right), \quad (22)$$

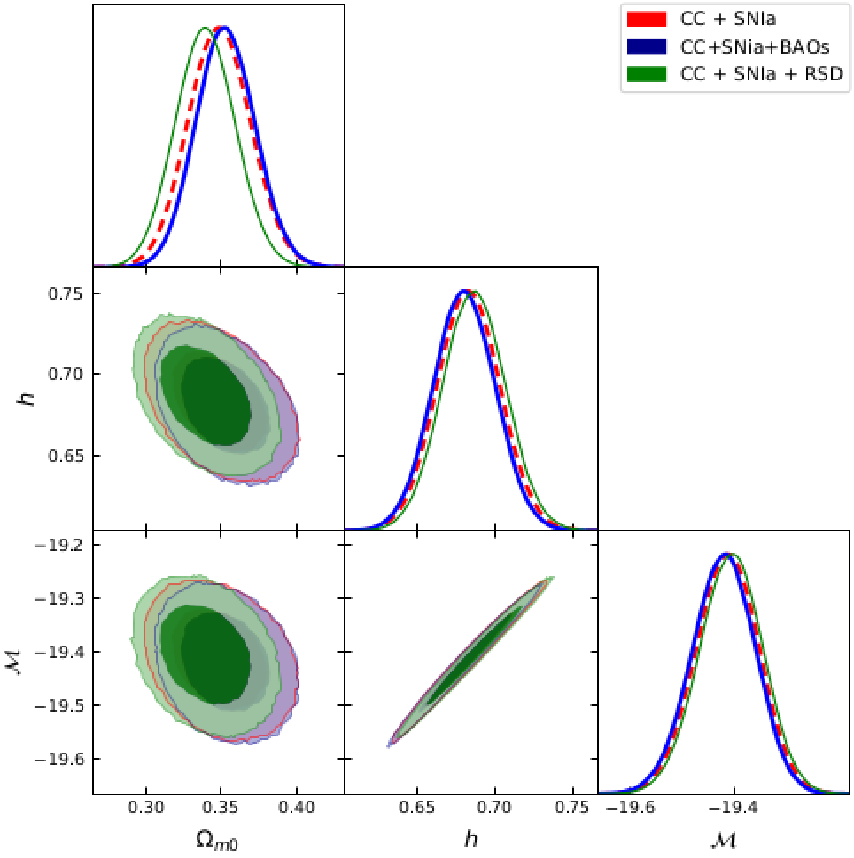
$$v' = -\mathcal{H}\left(1 - c_s^2\right) v + \frac{\delta p}{\rho + p} + \phi\,. \quad (23)$$

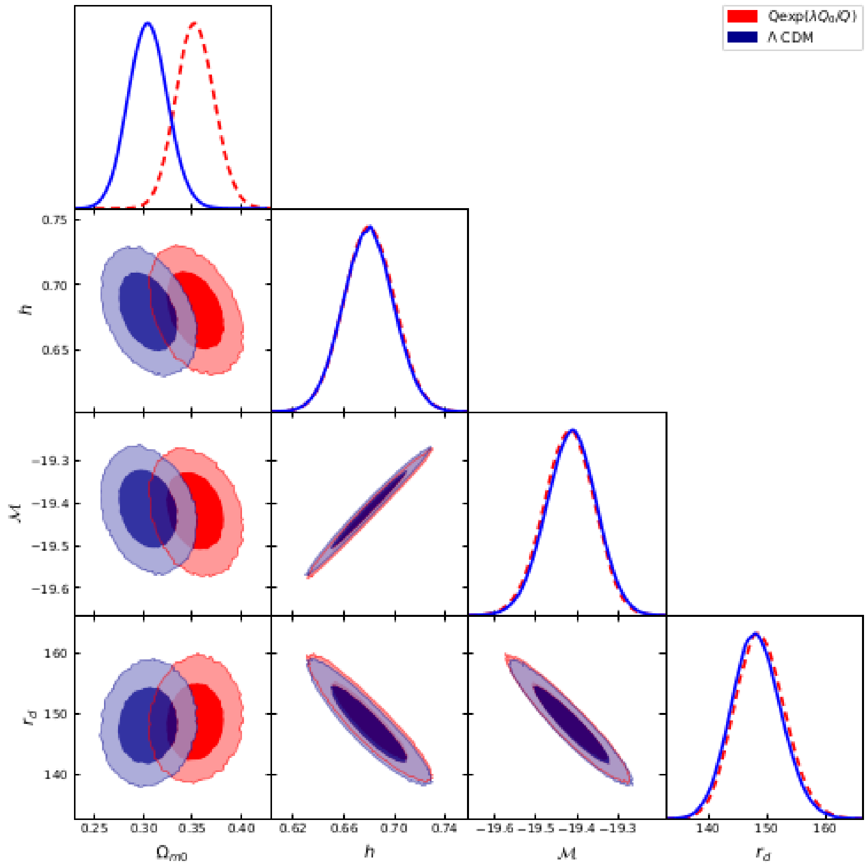
$$\begin{aligned} &- \; f_{QQ}\mathcal{H}\left[2\mathcal{H}\varphi' + \left(\mathcal{H}' + \mathcal{H}^2\right)\phi + \left(\mathcal{H}' - \mathcal{H}^2\right)(\psi - B')\right] \\ &- \; \left[f_{QQ}\left(\mathcal{H}'^2 + \mathcal{H}\mathcal{H}'' - 3\mathcal{H}^2\mathcal{H}' - \frac{1}{3}\mathcal{H}^2 k^2\right)\right. \\ &+ \; \left.\frac{df_{QQ}}{d\tau}(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}\right] B = 0\,, \quad (24) \end{aligned}$$

$$\delta'' + \mathcal{H}\delta' = \frac{4\pi G\rho}{f_Q}\delta\,, \quad (30)$$

$$G_{eff} \equiv \frac{G}{f_Q}\,, \quad (31)$$

f(Q) cosmology





Testing GR and Modified Gravity

- Solar-System data
- Galaxy data
- Galaxy-cluster data
- Cosmological data (SNIa, BAO,CMB,CC, LSS)
- Early Universe (Inflation, Baryogenesis, BBN)
- Black-Hole-shadow data
- Gravitational-wave data (multi-messenger astronomy)

Cosmology-background

- Homogeneity and isotropy: $ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$

- Background evolution (Friedmann equations) in flat space

$$H^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_{DE}\right)$$
$$\dot{H} = -4\pi G\left(\rho_m + p_m + \rho_{DE} + p_{DE}\right),$$

(the effective DE sector can be either Λ or any possible modification)

- One must obtain a H(z) and Ωm(z) and wDE(z) in agreement with observations (SNIa, BAO, CMB shift parameter, H(z) etc)

Cosmology-perturbations

- Perturbation evolution: $\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\rm eff}\,\rho\,\delta \approx 0$ where $\delta \equiv \delta\rho/\rho$

  where $G_{\rm eff}(z,k)$ is the effective Newton's constant, given by

  $$\nabla^2\phi \approx 4\pi G_{\rm eff}\rho\,\delta,$$

  under the scalar metric perturbation $ds^2 = -(1+2\phi)dt^2 + a^2(1-2\psi)d\vec{x}^{\,2}$

- Hence: $$\delta'' + \left(\frac{(H^2)'}{2\,H^2} - \frac{1}{1+z}\right)\delta' \approx \frac{3}{2}(1+z)\frac{H_0^2}{H^2}\frac{G_{\rm eff}(z,k)}{G_N}\,\Omega_{0m}\delta$$

  with $f(a) = \frac{dln\delta}{dlna}$ the growth rate, with $f(a) = \Omega_{\rm m}(a)^{\gamma(a)}$ and $\Omega_{\rm m}(a) \equiv \frac{\Omega_{0m}\,a^{-3}}{H(a)^2/H_0^2}$

- One can define the observable: $\boxed{f\sigma_8(a) \equiv f(a)\cdot\sigma(a) = \frac{\sigma_8}{\delta(1)}\,a\,\delta'(a)}$

  with $\sigma(a) = \sigma_8\frac{\delta(a)}{\delta 1}$ the z-dependent rms fluctuations of the linear density field within spheres of radius $R = 8h^{-1}\mathrm{Mpc}$, and σ8 its value today.

Black-hole shadow – EHT datasets

- In the center of M87, mass 7.000.000.000 Solar masses

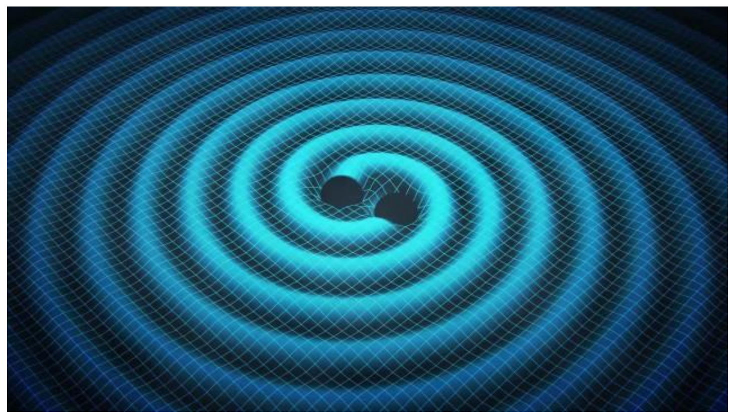
$$\delta = (42 \pm 3)\,\mu\text{arcsec},$$
$$M = (6.5 \pm 0.9) \times 10^9\, M_\odot,$$
$$D = 16.8^{+0.8}_{-0.7}\ \text{Mpc},$$

$$d_{M87*} \equiv \frac{D\delta}{M} \approx 11.0 \pm 1.5\,.$$

- Black-hole solutions -> effective potential -> photon sphere -> Black-hole shadow

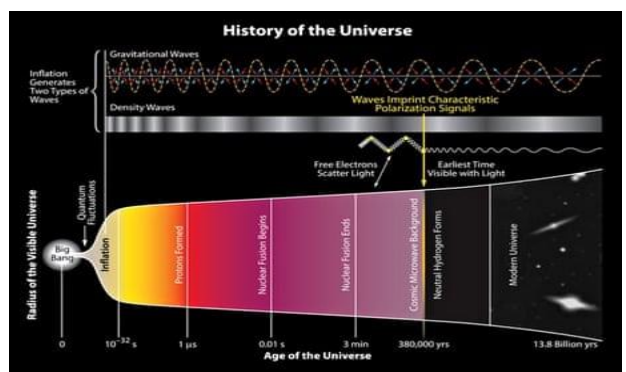
Gravitational waves

- The GWs are the tensor perturbations of the metric. Predicted in 1915, first observed in 2015. First astronomical observation ever, not related to E/M (or neutrinos).
- GWs from mergers:



[Abbott et al, LIGO Virgo PRL 116]

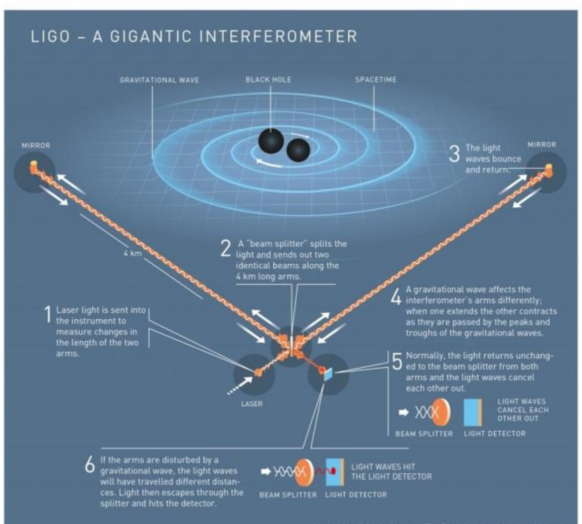
- Primordial GWs:

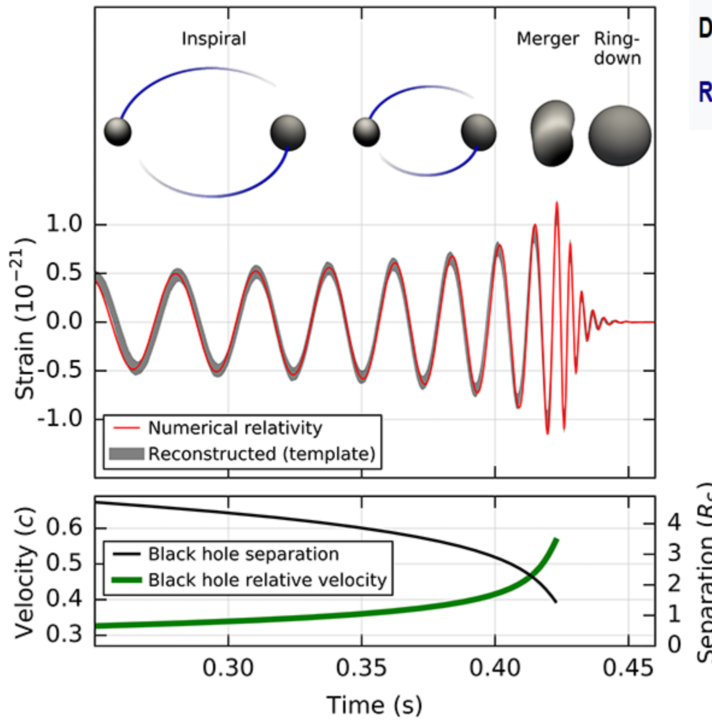


Gravitational waves

- $\color{red}{\text{GW150914}}$: Two black holes with 36^{+5}_{-4} M⊙ and 29^{+4}_{-4} M⊙, resulting in a 62^{+4}_{-4} M⊙ black hole

Louisiana.
Washington
4km
10^{-18}m





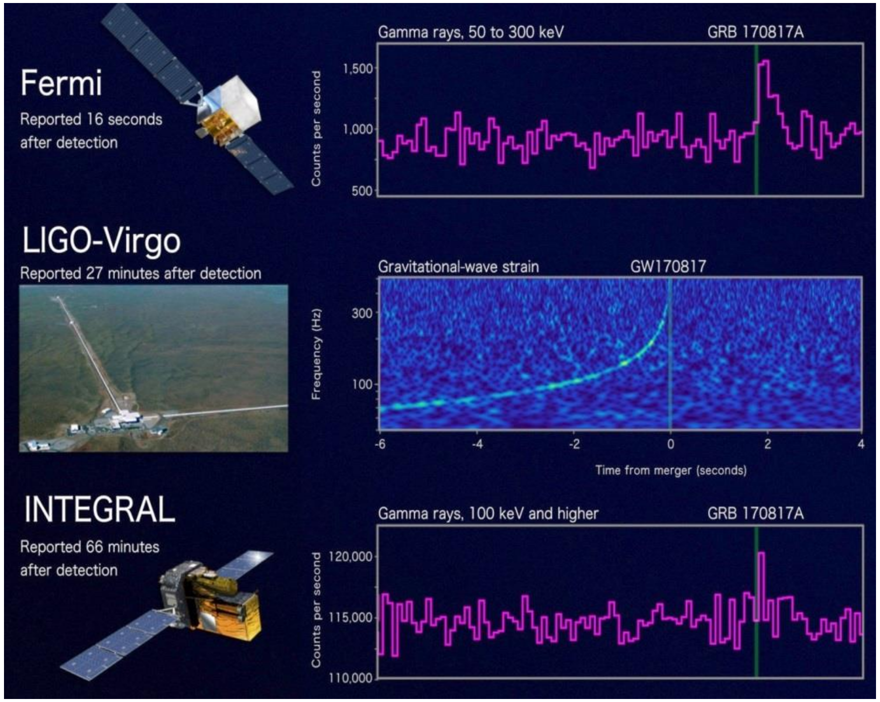
Distance	410^{+160}_{-180} Mpc
Redshift	$0.093^{+0.030}_{-0.036}$

[Abbott et al, LIGO Virgo PRL 116]

2017 Nobel Price in Physics

Gravitational waves

- GW170817: Two neutron stars, distance 40 Mpc, redshift 0.0099
- GRB170817A: The Electromagnetic counterpart.



- The era of multi-messenger astronomy begins!

Gravitational waves

- In case of GWs from black hole mergers we know their properties at the moment of detection, and their direction (in case of three detectors). Assuming GR and ΛCDM we can extract their speed, distance, and properties at the moment of emission.

Gravitational waves

- In case of GWs from black hole mergers we know their properties at the moment of detection, and their direction (in case of three detectors). Assuming GR and ΛCDM we can extract their speed, distance, and properties at the moment of emission.

- In case of GWs from neutron star mergers, and their E/M counterpart, we know their properties at the moment of detection and their direction, but using the implied physics from the E/M information we can extract their speed, distance and properties at the moment of emission, independently of the underlying gravitational theory and cosmological scenario.

- Great tool for testing General Relativity and cosmological scenarios!

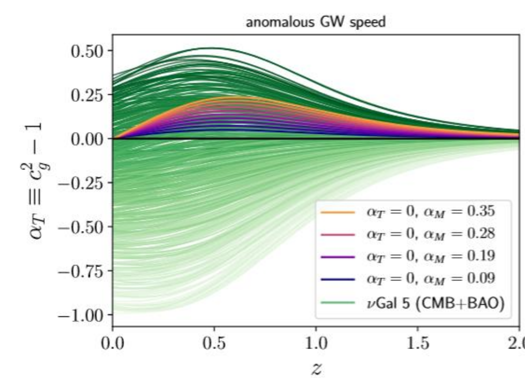
Gravitational waves

- An immediate result: The speed of GWs is equal to the speed of light!

  GW170817 time delay $1.74 \pm 0.05\mathrm{s}$ constrains: $\boxed{-3\cdot 10^{-15} \leq c_g/c - 1 \leq 7\cdot 10^{-16}}$

- Excludes a large number of theories that were consistent with other data!

	$c_g = c$	$c_g \neq c$
Horndeski	General Relativity	quartic/quintic Galileons [13, 14]
	quintessence/k-essence [46]	Fab Four [15]
	Brans-Dicke/$f(R)$ [47, 48]	de Sitter Horndeski [49]
	Kinetic Gravity Braiding [50]	$G_{\mu\nu}\phi^\mu\phi^\nu$ [51], $f(\phi)$·Gauss-Bonnet [52]
beyond H.	Derivative Conformal (19) [17]	quartic/quintic GLPV [18]
	Disformal Tuning (21)	quadratic DHOST [20] with $A_1 \neq 0$
	quadratic DHOST with $A_1 = 0$	cubic DHOST [23]
	Viable after GW170817	Non-viable after GW170817



[Ezquiaga, Zumalacarregui PRL 119]

Gravitational waves

- For tensor perturbations: $$g_{00} = -1\,, \quad g_{0i} = 0\,,$$ $$g_{ij} = a^2\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h_{kj}\right)$$

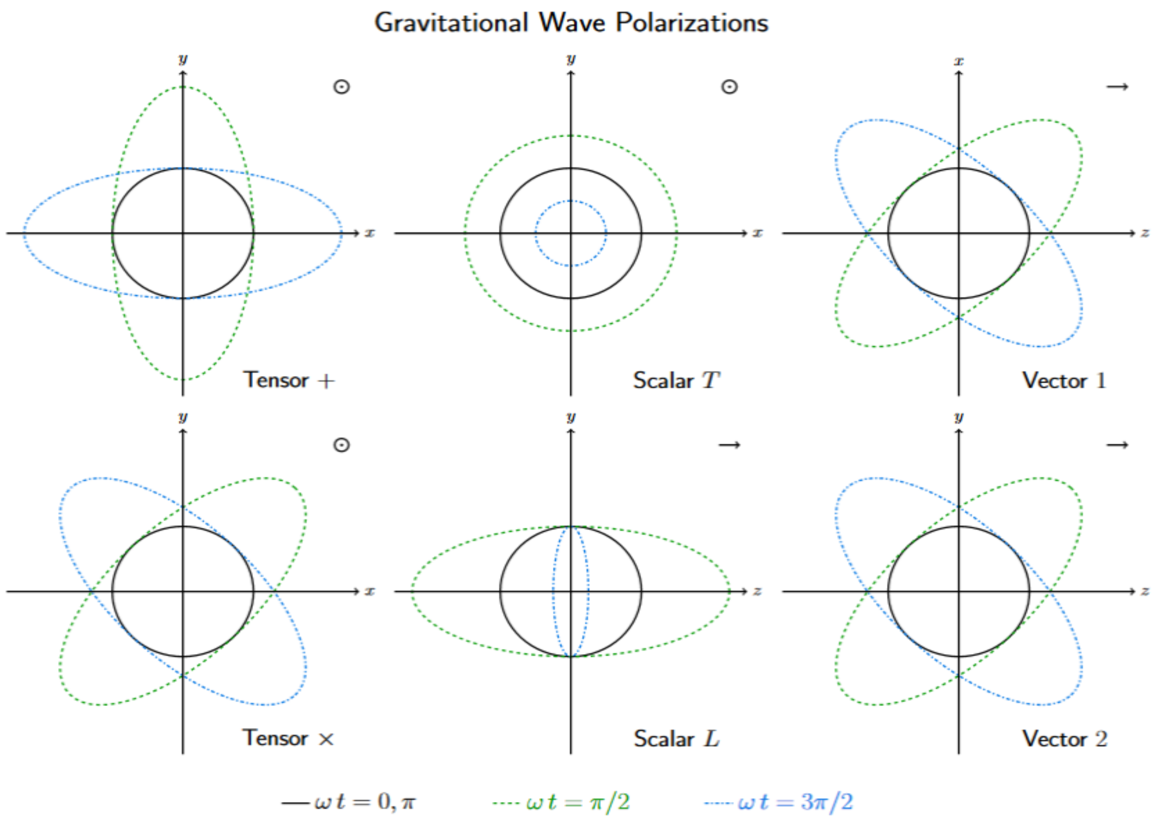
$$\boxed{\ddot{h}_{ij} + (3+\alpha_M)\dot{h}_{ij} + (1+\alpha_T)\frac{k^2}{a^2}h_{ij} = 0}$$

$$\alpha_M = \frac{d\log(M_*^2)}{d\log a} \qquad\qquad c_g^2 = (1+\alpha_T)$$

- $$h_{\rm GW} \sim h_{\rm GR} \quad \underbrace{e^{-\frac{1}{2}\int \nu\mathcal{H}d\eta}}_{\text{Affects amplitude}} \quad \underbrace{e^{ik\int(\alpha_T + a^2m^2/k^2)^{1/2}d\eta}}_{\text{Affects phase}}$$

Gravitational waves

- Polarizations:



Gravitational waves in modified gravity

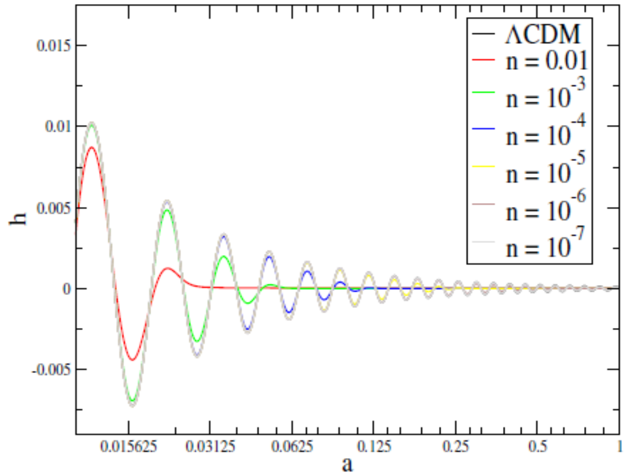
- Gw's propagation at cosmological scales: $h = e^{-\mathcal{D}} e^{-ik\Delta T} h_{GR}$

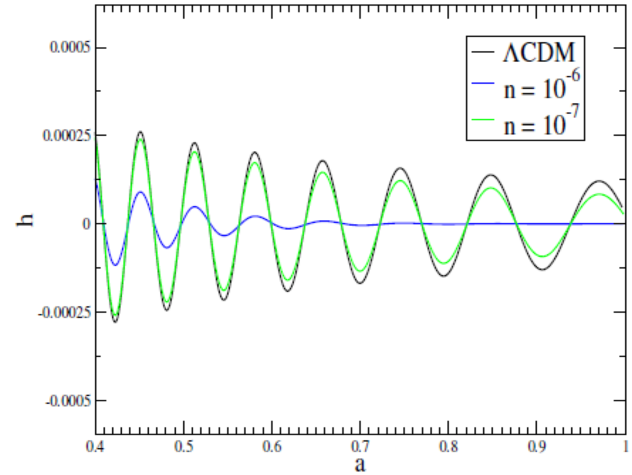
$$\mathcal{D} = \frac{1}{2}\int \nu \mathcal{H} d\tau' \quad \text{(affects amplitude)} \qquad \Delta T = \int \left(1 - c_T - \frac{a^2\mu^2}{2k^2}\right) d\tau' \quad \text{(affects phase)}$$

- In f(T) gravity:

$$\ddot{h}_{ij} + 3H(1-\beta_T)\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 0$$

$$\beta_T \equiv -\frac{\dot{f}_T}{3Hf_T}$$





Gravitational Waves in Modified Teleparallel Theories

- $$S = \frac{1}{16\pi G}\int d^4x\, e\, f(T,B) + \int d^4x\, e\, \mathcal{L}_m$$ $$R = -T - 2\nabla^\mu T^\nu{}_{\mu\nu}$$

$$\boxed{\begin{aligned}&- f_T G_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) f_B \\ &+ \frac{1}{2} g_{\mu\nu}\left(f_B B + f_T T - f\right) \\ &+ 2S_\nu{}^\alpha{}_\mu \partial_\alpha \left(f_T + f_B\right) = 8\pi G\, \Theta_{\mu\nu}\end{aligned}}$$

- $$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + \mathcal{O}\left(h^{(2)}_{\mu\nu}\right)$$

$$h^{(1)}_{\mu\nu} = \begin{pmatrix} -2A\exp(ik_\mu x^\mu) - \frac{f^{(0)}_{BB}B^{(1)}}{f^{(0)}_T} & B_1\exp(ik_\mu x^\mu) & B_2\exp(ik_\mu x^\mu) & -2A\exp(ik_\mu x^\mu) \\ B_1\exp(ik_\mu x^\mu) & h_+ + \frac{f^{(0)}_{BB}B^{(1)}}{f^{(0)}_T} & h_\times & B_1\exp(ik_\mu x^\mu) \\ B_2\exp(ik_\mu x^\mu) & h_\times & -h_+ + \frac{f^{(0)}_{BB}B^{(1)}}{f^{(0)}_T} & B_2\exp(ik_\mu x^\mu) \\ -2A\exp(ik_\mu x^\mu) & B_1\exp(ik_\mu x^\mu) & B_2\exp(ik_\mu x^\mu) & -2A\exp(ik_\mu x^\mu) + \frac{f^{(0)}_{BB}B^{(1)}}{f^{(0)}_T} \end{pmatrix}$$

[Farrugia, Said, Gakis, Saridakis, PRD 97]

The Effective Field Theory (EFT) approach

- The EFT approach allows to ignore the details of the underlying theory and write an action for the perturbations around a time-dependent background solution.
- One can systematically analyze the perturbations separately from the background evolution. [Arkani-Hamed, Cheng JHEP0405 (2004)]

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$$
\begin{aligned}
S = \int d^4x \Big\{ \sqrt{-g} \Big[& \frac{M_P^2}{2}\Psi(t)R - \Lambda(t) - b(t)g^{00} && \text{<- background} \\
& + M_2^4(\delta g^{00})^2 - \bar{m}_1^3 \delta g^{00}\delta K - \bar{M}_2^2 \delta K^2 - \bar{M}_3^2 \delta K^\nu_\mu \delta K^\mu_\nu && \text{<- linear evolution of perturbations} \\
& + m_2^2 h^{\mu\nu}\partial_\mu g^{00}\partial_\nu g^{00} + \lambda_1 \delta R^2 + \lambda_2 \delta R_{\mu\nu}\delta R^{\mu\nu} + \mu_1^2 \delta g^{00}\delta R \Big] && \text{<- linear evolution of perturbations} \\
& + \gamma_1 C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \gamma_2 \epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda} && \text{<- linear evolution of perturbations} \\
& + \sqrt{-g}\Big[\frac{M_3^4}{3}(\delta g^{00})^3 - \bar{m}_2^3(\delta g^{00})^2\delta K + \ldots\Big]\Big\} , && \text{<- 2}^{\text{nd}}\text{-order evolution of perturbations}
\end{aligned}
$$

The functions Ψ(t), Λ(t), b(t), are determined by the background solution

[Gubitosi, Piazza, Vernizzi, JCAP1302]

The (EFT) approach to torsional gravity

- Application of the EFT approach to torsional gravity leads to include terms:
- i) Invariant under 4D diffeomorphisms: e.g. R,T multiplied by functions of time.
- ii) Invariant under spatial diffeomorphisms: e.g. $g^{00},\ R^{00}$ and T^{0}
- ii) Invariant under spatial diffeomorphisms: e.g. ${}^{(3)}R_{\mu\nu\rho\sigma}$, ${}^{(3)}T^{\rho}{}_{\mu\nu}$, $K_{\mu\nu}$, $\hat{K}_{\mu\nu}$

  the extrinsic torsion is defined as

$$\hat{K}_{\mu\nu} \equiv h^{\sigma}_{\mu}\hat{\nabla}_{\sigma}n_{\nu} = K_{\mu\nu} - \mathcal{K}^{\lambda}{}_{\nu\mu}n_{\lambda} + n_{\mu}\frac{1}{g^{00}}T^{00}{}_{\nu}\ ,$$

with n_{μ} the orthogonal to t=cont. surfaces unitary vector $n_{\mu} = \frac{\delta^{0}_{\mu}}{\sqrt{-g^{00}}}$

[Cai, Li, Saridakis, Xue, PRD 97], [Li, Cai, Cai, Saridakis, JCAP18]

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with n_{μ} the orthogonal to t=cont. surfaces unitary vector $n_{\mu} = \frac{\delta^{0}_{\mu}}{\sqrt{-g^{00}}}$

Using the projection operator h^{μ}_{ν} we can express ${}^{(3)}R_{\mu\nu\rho\sigma} = h^{\alpha}_{\mu}h^{\beta}_{\nu}h^{\gamma}_{\rho}h^{\delta}_{\sigma}R_{\alpha\beta\gamma\delta} - K_{\mu\rho}K_{\nu\sigma} + K_{\nu\rho}K_{\mu\sigma}$

$$h^{d}_{a}h^{c}_{b}h^{f}_{e}T^{e}{}_{dc} = {}^{(3)}T^{f}{}_{ab}$$

[Cai, Li, Saridakis, Xue, PRD 97], [Li, Cai, Cai, Saridakis, JCAP18]

The (EFT) approach to torsional gravity

- We perturb the previous tensors, and we finally obtain:

$$
\begin{aligned}
R^{(0)}_{\mu\nu\rho\sigma} &= f_1(t) g_{\mu\rho} g_{\nu\sigma} + f_2(t) g_{\mu\rho} n_\nu n_\sigma + f_3(t) g_{\mu\sigma} g_{\nu\rho} \\
&\quad + f_4(t) g_{\mu\sigma} n_\nu n_\rho + f_5(t) g_{\nu\sigma} n_\mu n_\rho \\
&\quad + f_6(t) g_{\nu\rho} n_\mu n_\sigma , \\
T^{(0)}_{\rho\mu\nu} &= g_1(t) g_{\rho\nu} n_\mu + g_2(t) g_{\rho\mu} n_\nu , \\
K^{(0)}_{\mu\nu} &= f_7(t) g_{\mu\nu} + f_8(t) n_\mu n_\nu , \\
\hat{K}^{(0)}_{\mu\nu} &= 0 \ .
\end{aligned}
$$

where the time-dependent functions are determined by the background solution.

[Cai, Li, Saridakis, Xue, PRD 97], [Li, Cai, Cai, Saridakis, JCAP18]

The (EFT) approach to torsional gravity

- Finally, the EFT action of torsional gravity becomes:

$$S = \int d^4x\sqrt{-g}\Big[\frac{M_P^2}{2}\Psi(t)R - \Lambda(t) - b(t)g^{00} + \frac{M_P^2}{2}d(t)T^0\Big]$$
$$+S^{(2)} \ ,$$

- The perturbation part contains:

  i) Terms present in curvature EFT action

  ii) Pure torsion terms such as δT^2 , $\delta T^0\delta T^0$ and $\delta T^{\rho\mu\nu}\delta T_{\rho\mu\nu}$

  iii) Terms that mix curvature and torsion, such as $\delta T\delta R$, $\delta g^{00}\delta T$, $\delta g^{00}\delta T^0$ and $\delta K\delta T^0$

[Cai, Li, Saridakis, Xue, PRD 97], [Li, Cai, Cai, Saridakis, JCAP18]

The (EFT) approach to f(T) gravity: Background

- For the case of f(T) gravity, at the background level, we have:

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{-g} \big[- f_T(T^{(0)}) R + 2 \dot{f}_T(T^{(0)}) T^{(0)} - T^{(0)} f_T(T^{(0)}) + f(T^{(0)}) \big]$$

where by comparison:

$$\Psi(t) = -f_T(T^{(0)}) \ ,$$
$$\Lambda(t) = \frac{M_P^2}{2} \left[T^{(0)} f_T(T^{(0)}) - f(T^{(0)}) \right] \ ,$$
$$d(t) = -2 \dot{f}_T(T^{(0)}) \ ,$$
$$b(t) = 0 \ .$$

- Performing variation we obtain the background equations of motion (Friedmann Eqs):

$$b(t) = M_P^2 \Psi \left(-\dot{H} - \frac{\ddot{\Psi}}{2\Psi} + \frac{H\dot{\Psi}}{2\Psi} - \frac{\dot{d}}{4\Psi} + \frac{3Hd}{4\Psi} \right) - \frac{1}{2}(\rho_m + p_m),$$

$$\Lambda(t) = M_P^2 \Psi \left(3H^2 + \frac{5H\dot{\Psi}}{2\Psi} + \dot{H} + \frac{\ddot{\Psi}}{2\Psi} + \frac{\dot{d}}{4\Psi} + \frac{3Hd}{4\Psi} \right) - \frac{1}{2}(\rho_m - p_m),$$

[Li, Cai, Cai, Saridakis, JCAP18]

The (EFT) approach to f(T) gravity: Tensor Perturbations

- For tensor perturbations: $g_{00} = -1\ , \quad g_{0i} = 0\ ,$

$$g_{ij} = a^2\big(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h_{kj}\big)$$

i.e.

$$\bar{e}^0_\mu = \delta^0_\mu\ ,$$
$$\bar{e}^a_\mu = a\delta^a_\mu + \frac{a}{2}\delta^i_\mu\delta^{aj}h_{ij} + \frac{a}{8}\delta^i_\mu\delta^{ja}h_{ik}h_{kj}\ ,$$
$$\bar{e}^\mu_0 = \delta^\mu_0\ ,$$
$$\bar{e}^\mu_a = \frac{1}{a}\delta^\mu_a - \frac{1}{2a}\delta^{\mu i}\delta^j_a h_{ij} + \frac{1}{8a}\delta^{i\mu}\delta^j_a h_{ik}h_{kj}$$

- We obtain:

$$^{(3)}R \approx -\frac{1}{4}a^{-2}(\partial_i h_{kl}\partial_i h_{kl})\ ,$$
$$K^{ij}K_{ij} \approx 3H^2 + \frac{1}{4}\dot{h}_{ij}\dot{h}_{ij}\ ,$$
$$K \approx 3H\ ,$$
$$\boxed{T = T^{(0)} + O(h^2) = 6H^2 + O(h^2)}$$

- And finally:

$$S = \frac{M_P^2}{2}\int d^4x\sqrt{-g}\Big[\frac{f_T}{4}\big(a^{-2}\vec{\nabla}h_{ij}\cdot\vec{\nabla}h_{ij} - \dot{h}_{ij}\dot{h}_{ij}\big) + 6H^2 f_T - 12H\dot{f}_T - T^{(0)}f_T + f(T^{(0)})\Big]$$

[Cai, Li, Saridakis, Xue, PRD 97]

The (EFT) approach to f(T) gravity: Scalar Perturbations

- For scalar perturbations:

$$g_{00} = -1 - 2\phi \ , \quad g_{0i} = 0 \ , \quad g_{ij} = a^2[(1-2\psi)\delta_{ij} + \partial_i\partial_j F]$$

i.e

$$e^0_\mu = \delta^0_\mu + \delta^0_\mu \phi + a\delta^i_\mu \partial_i \chi \ ,$$
$$e^a_\mu = a\delta^i_\mu \delta^a_i + \delta^0_\mu \delta^a_i \partial^i \mathcal{E} + a\delta^i_\mu \delta^a_j \left[\epsilon_{ijk}\partial_k\sigma - \psi\delta_{ij} + \frac{1}{2}\partial_i\partial_j F\right]$$

- So $T^0 = g^{0\mu}T^\nu{}_{\mu\nu} = -3H + 6H\phi + 3\dot{\psi} - 6H\phi^2 - 6\dot{\psi}\phi$

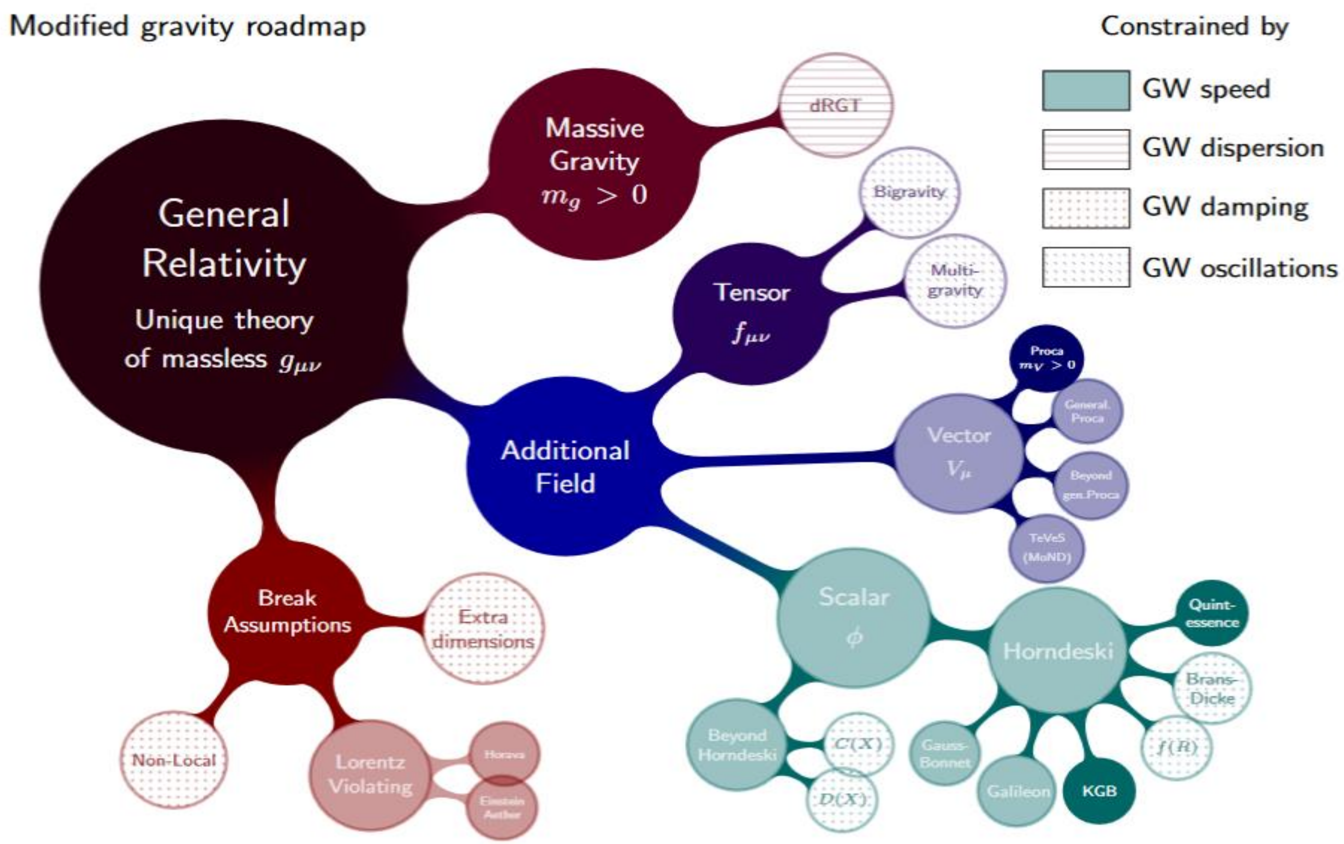
$$+ \frac{1}{a}\partial_i\partial_i\chi - \frac{1}{2a}\partial_i\phi\partial_i\chi - \frac{3}{2a}\phi\partial_i\partial_i\chi - \frac{1}{2a}\partial_i\psi\partial_i\chi + \frac{1}{2a}\psi\partial_i\partial_i\chi$$

- Thus:

$$\boxed{S = \int d^4x \Big[\frac{M_P^2}{2}\Big(-2af_T\partial_i\psi\partial_i\psi + 4af_T\partial_i\phi\partial_i\psi + 4a^2\dot{f}_T\partial_i\psi\partial_i\chi + 4\dot{f}_T a^2 H\partial_i\pi\partial_i\chi\Big) + a^3M^2\pi^2 - a^3\phi\delta\rho_m\Big]}$$

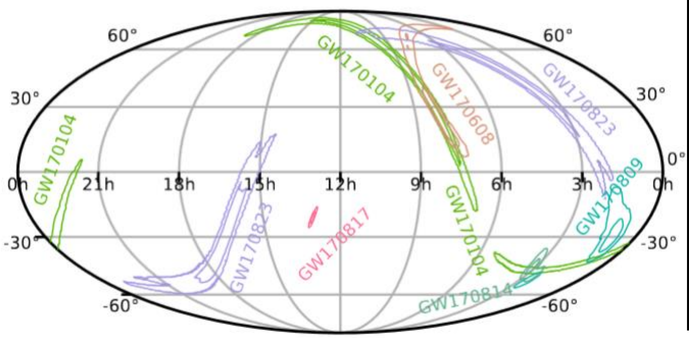
[Li, Cai, Cai, Saridakis, JCAP18]

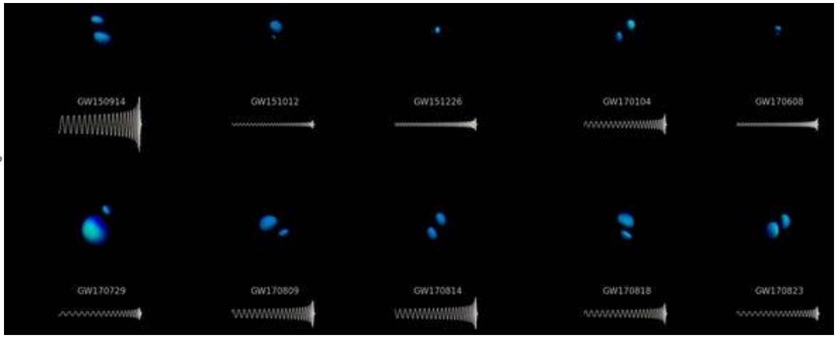
Gravitational waves and Modified Gravity



Gravitational waves - Observations

- Observations: 80up to now (69 BH-BH, 4 NS-NS, 3 NS-BH, 4 uncertain, 19-85 Msun, 320-2800 MPc)





Designation
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151226+03:38:53UTC
151012+09:54:43UTC
151019+00:23:16UTC
150928+10:49:00UTC
151218+18:30:58UTC
160103+05:48:36UTC
151202+01:18:13UTC
160104+03:51:51UTC
151213+00:12:20UTC
150923+07:10:59UTC
151029+13:34:39UTC
151206+14:19:29UTC
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151012+06:30:45UTC
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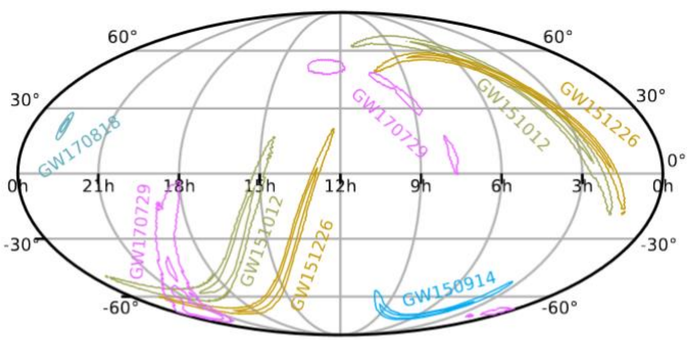
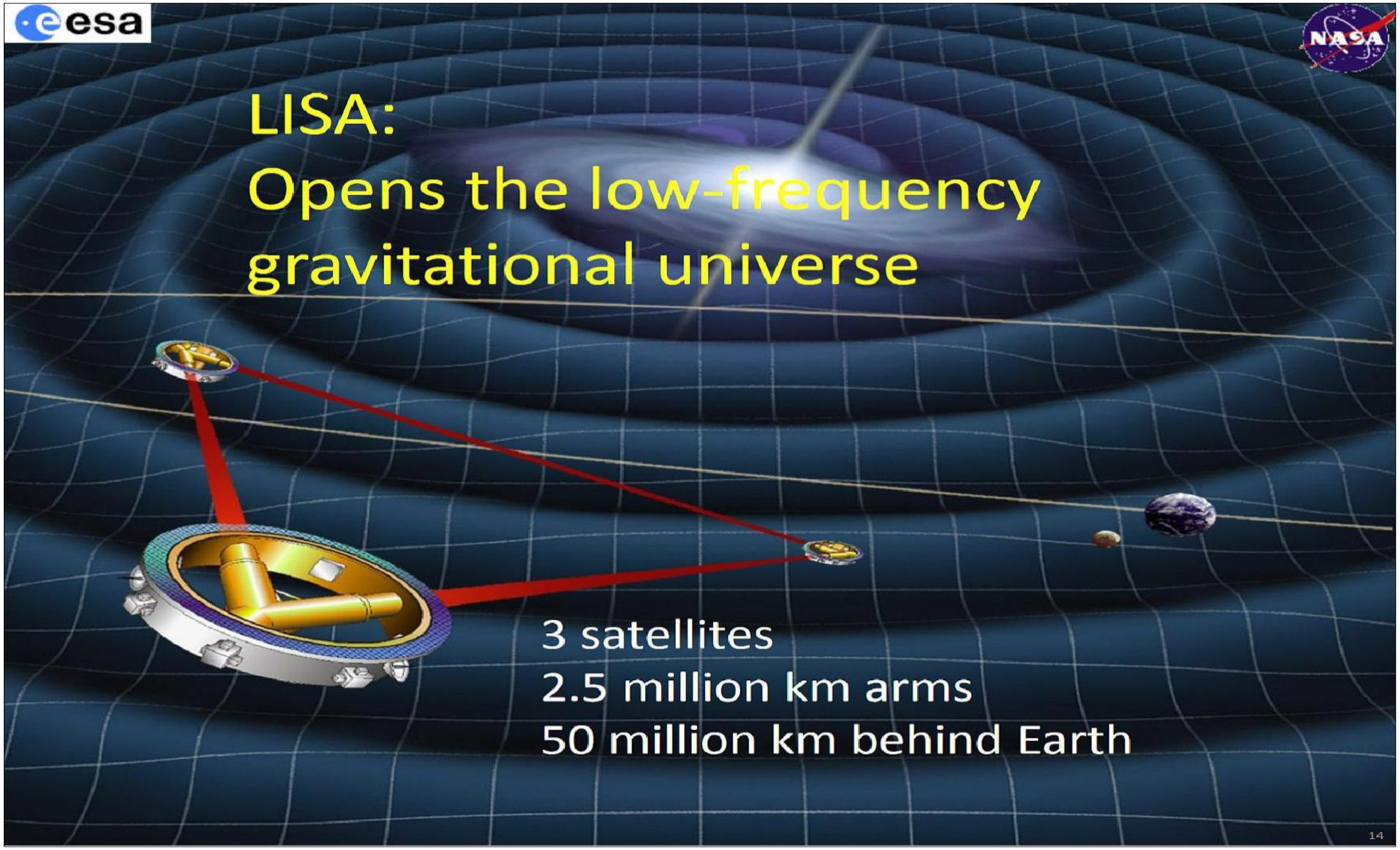


Event	$m_1/M_\odot$	$m_2/M_\odot$	$\mathcal{M}/M_\odot$
GW150914	$35.6^{+4.8}_{-3.0}$	$30.6^{+3.0}_{-4.4}$	$28.6^{+1.6}_{-1.5}$
GW151012	$23.3^{+14.0}_{-5.5}$	$13.6^{+4.1}_{-4.8}$	$15.2^{+2.0}_{-1.1}$
GW151226	$13.7^{+8.8}_{-3.2}$	$7.7^{+2.2}_{-2.6}$	$8.9^{+0.3}_{-0.3}$
GW170104	$31.0^{+7.2}_{-5.6}$	$20.1^{+4.9}_{-4.5}$	$21.5^{+2.1}_{-1.7}$
GW170608	$10.9^{+5.3}_{-1.7}$	$7.6^{+1.3}_{-2.1}$	$7.9^{+0.2}_{-0.2}$
GW170729	$50.6^{+16.6}_{-10.2}$	$34.3^{+9.1}_{-10.1}$	$35.7^{+6.5}_{-4.7}$
GW170809	$35.2^{+8.3}_{-6.0}$	$23.8^{+5.2}_{-5.1}$	$25.0^{+2.1}_{-1.6}$
GW170814	$30.7^{+5.7}_{-3.0}$	$25.3^{+2.9}_{-4.1}$	$24.2^{+1.4}_{-1.1}$
GW170817	$1.46^{+0.12}_{-0.10}$	$1.27^{+0.09}_{-0.09}$	$1.186^{+0.001}_{-0.001}$
GW170818	$35.5^{+7.5}_{-4.7}$	$26.8^{+4.3}_{-5.2}$	$26.7^{+2.1}_{-1.7}$
GW170823	$39.6^{+10.0}_{-6.6}$	$29.4^{+6.3}_{-7.1}$	$29.3^{+4.2}_{-3.2}$

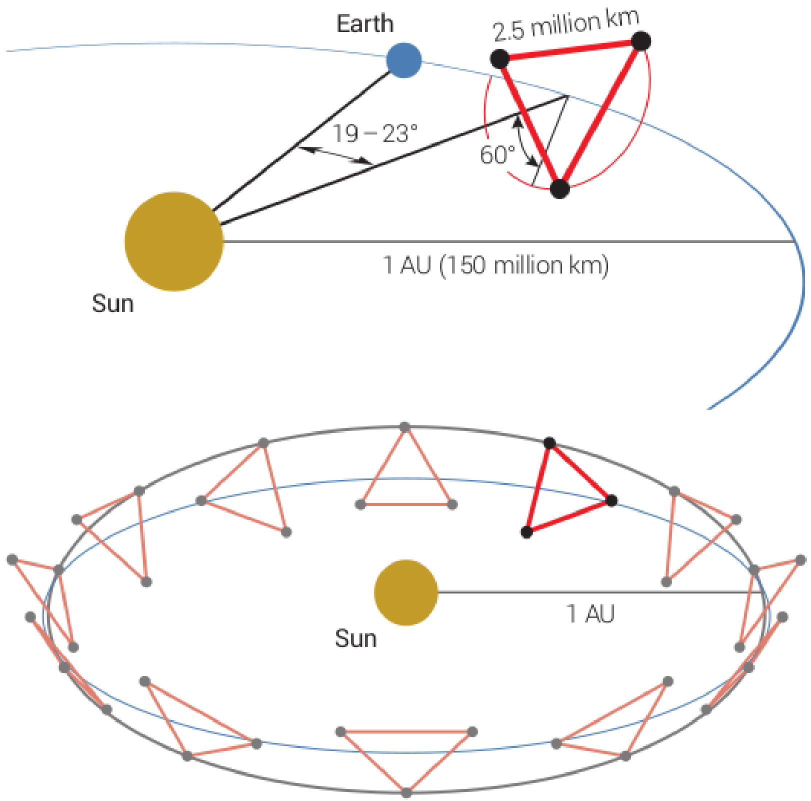
- Expectations: Many thousands in the next years

LISA

- LISA will be comprised by a constellation of three spacecrafts on heliocentric orbit. Each spacecraft will enclose two test masses kept in free-fall conditions. The differential distance between the test masses of far-away spacecrafts (across 2.5 × 10^6 km) will be monitored by means of laser interferometry.



Sensitivity



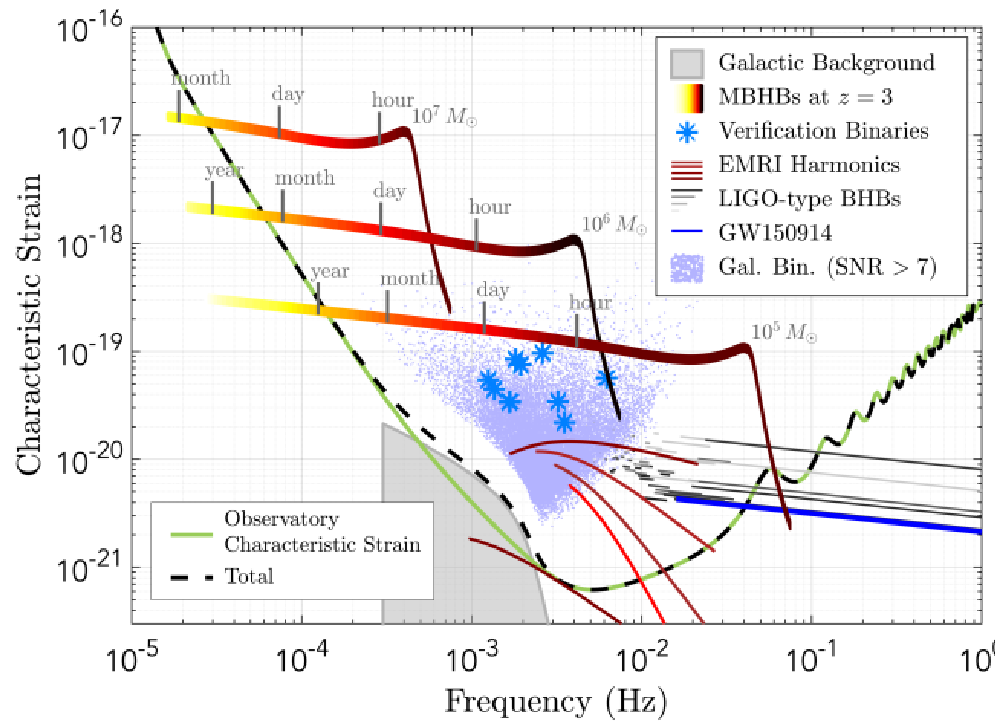
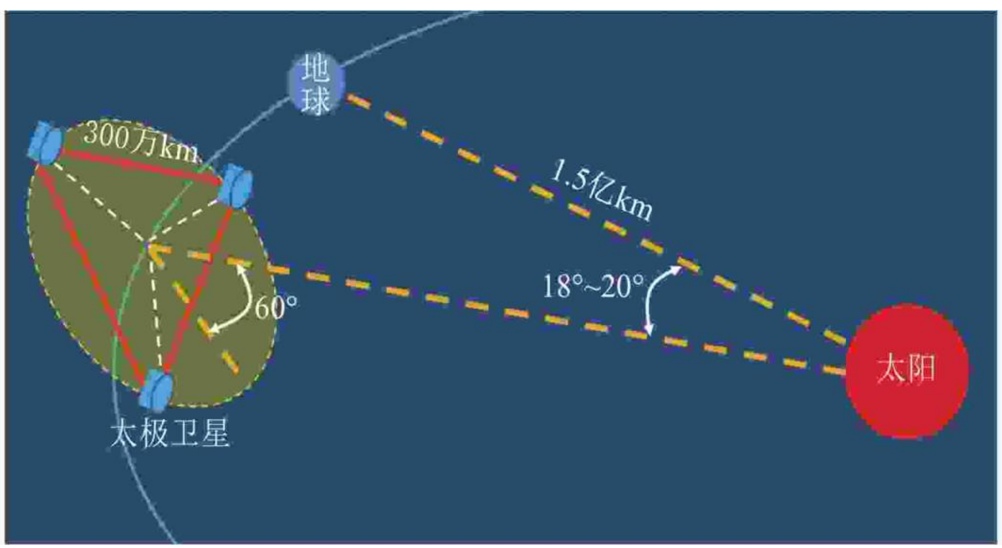
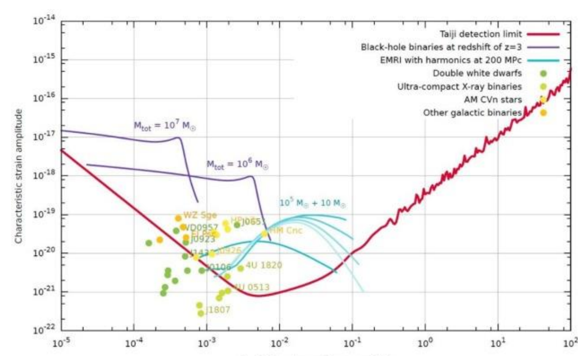


FIGURE 1 – *Left* : Cartoon of the ESA LISA constellation and the orbits designed for the mission. *Right* : Plot of the LISA sensitivity and the most probable sources that LISA can probe. We expect to measure supermassive and stellar-mass black hole binaries, as well as the ensemble signal originating from the ultra-compact binaries of our Galaxy (mostly Double White Dwarfs).

Taiji Program in Space

- The Taiji Program in Space, is a proposed Chinese satellite-based gravitational-wave observatory. It is scheduled for launch in 2033 to study gravitational waves. The program consists of a triangle of three spacecraft spacecrafts (across 3 × 10^6 km) orbiting the sun linked by laser interferometers.

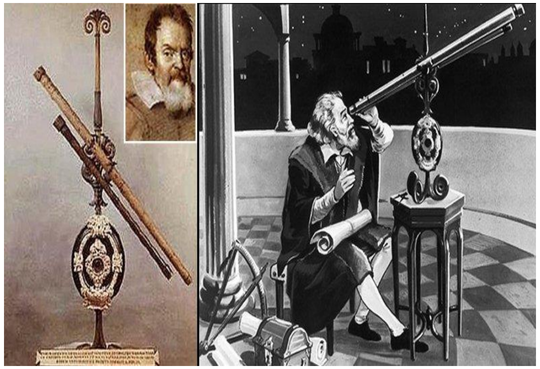




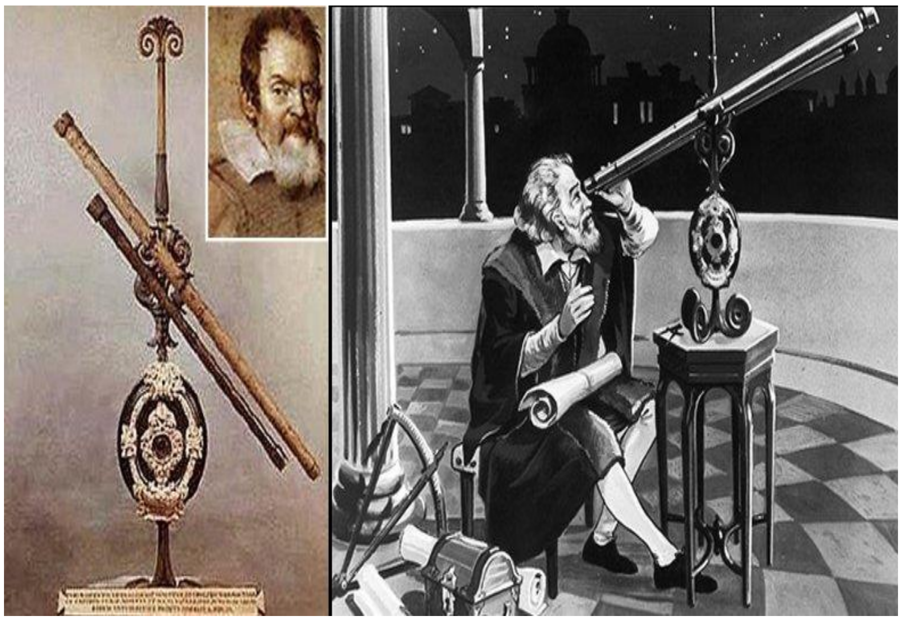
5000 years of observations
500 years of organized observations







Multi-messenger Astronomy Era!



EM observations: 400 years

GW observations: 6 years

- "There are the ones that invent occult fluids to understand the Laws of Nature. They come to conclusions, but they now run out into dreams and chimeras neglecting the true constitutions of the things...
  However there are those that from the simplest observation of Nature, they reproduce New Forces"...

From the Preface of PRINCIPIA (II edition) 1687
by Isaac Newton, written by Mr. Roger Cotes.

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THANK YOU!